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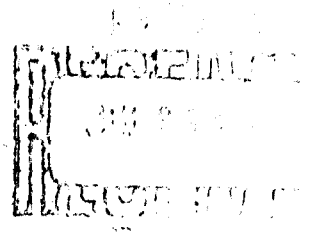
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A FORMAL SOLUTION TO MAXWELL'S  
EQUATIONS FOR GENERAL LINEAR MEDIA

by

Paul Diament



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FOR GENERAL LINEAR MEDIA

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## ABSTRACT

Maxwell's equations for linear media are reformulated through linear operator and generalized transform techniques into an equivalent matrix integral equation. An explicit formal solution to the equation is obtained recursively, providing a sequence of operations to be applied to the electrical parameters of the medium to yield the characteristic existence conditions, the set of normal modes, and the electromagnetic fields in response to given sources. The results are applicable to time-invariant, linear media which may be inhomogeneous, anisotropic, nonuniform, dissipative, dispersive, with any source distribution.

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## 1. INTRODUCTION

Electromagnetic phenomena are completely described mathematically by Maxwell's equations. The fundamental problem of electromagnetic theory is to transform the implicit description embodied in the equations into explicit formulations of these phenomena through some procedure for solving the equations. It is a widely recognized truism that Maxwell's equations are amenable to solution in only a few cases possessing rather special simplifying features. A less restricted type of problem is usually solved, if at all, by appeal to methods often specifically designed for that particular problem. There remains a wide class of problems that resist analysis. A general method of attacking such problems is developed in this work.

In typical problems that do yield to standard analysis, the geometry possesses considerable symmetry, the medium consists of only a few homogeneous regions, the coordinate system appropriate to the structure of the system is one of the few in which the wave equation is separable, and the boundaries are of such nature and shape that it becomes feasible to attempt a straightforward solution to Maxwell's equations. The procedure is then to write these equations and the associated boundary conditions for each region of the space and to solve each such set of partial differential equations by separation of variables. This introduces a number of physically meaningful separation constants and arbitrary coefficients. The individual solutions must then be made to fit the boundary conditions, particularly those arising at the interfaces between regions of differing electrical properties.



This produces an auxiliary set of equations, no longer differential but merely algebraic and transcendental, which, beside determining the arbitrary coefficients, provides a relation among the separation constants. This characteristic relation, which may be a dispersion relation for a propagation problem or a resonance condition for a cavity, is often more illuminating and useful than the explicit expressions for the field components themselves.

The class of problems for which solutions in closed form are obtainable in this way is quite small. If the problem is not sufficiently simple to permit the above direct attack, recourse is often had to some form of perturbation approach. A similar but simpler problem is first solved exactly and the set of normal modes so obtained is used as a basis for representing the solution to the perturbed case as a superposition of modes of the simple case. The extent to which each normal mode contributes to the expansion remains to be determined through the use of orthogonality properties enjoyed by these modes by virtue of the symmetry of the unperturbed configuration. Green's functions, the use of which greatly extends the class of problems solvable by these methods, are usually found in the same way.

The mathematical elegance associated with the closed forms of solution and of the expressions in terms of more or less well-known and well-tabulated functions often turns out to be an illusory advantage of the above methods of solution. The characteristic equations, which provide in algebraic form implicit information about physical effects such as dispersion, scattering, or resonance, may require exhaustive studies of the mathematical properties of the constituent functions, as is also the case when the distribution and flow of electromagnetic

energy is to be computed from the field components. These tasks, though they are facilitated by a host of recursion and orthogonality properties possessed by these functions, must often be relegated to automatic computers, which must in turn rely upon the power series expansions of the elementary functions involved.

Abstracting the significant features of these general and powerful methods of solution, the following observations may be made. Firstly, the linearity of the equations underlies the methods, heavy reliance being placed upon the superposition principle. Secondly, the search for closed forms of solutions may be self-defeating in that their complexity may inhibit their interpretation and application. Thirdly, the recursive, reciprocal, and orthogonality properties of the solutions so obtained are actually properties of the normal modes associated with a given configuration of materials in the medium, not just of the various elementary functions involved, which are after all defined as solutions of the separated wave equations. Fourthly, the algebraic characteristic equations effectively supplant the original differential equations, are equivalent to the Maxwell equations specialized to the particular medium, and contain essentially the same physical information. Finally, these characteristic equations are entirely determined by the electrical constitution of the medium and should be obtainable without reference to the associated electromagnetic fields.

These principles are the foundation for the method of solution to be developed. Primarily, the goal is to reformulate the contents of Maxwell's equations so as to lead from a description of the spatial distribution of matter in the medium directly to a single characteristic

equation which prescribes the conditions for the existence of a mode. Secondly, a formulation is sought which will yield the complete set of normal modes for any linear medium. The form in which the results are to be expressed is to be left arbitrary, the choice to be made in advance on the basis of convenience and practicality.

In view of the nondifferential nature of the characteristic equations whose content is equivalent to that of the original equations, the present approach seeks to eliminate the differential character of the equations at the outset. This can be accomplished by means of a transformation from coordinate space to some new domain in which differentiation is replaced by other operations which also absorb the associated boundary conditions. Full advantage is taken of the linearity of the problem, in assuming representations of unknown functions as general summations, in expressing the response to several excitations as the superposition of individual responses, and as the essence of the reformulation in terms of linear operator techniques from which general properties of electromagnetic waves may be deduced.

A guiding principle in the reformulation of Maxwell's equations is that no information which could be extracted from the original equations be rendered unobtainable from the new machinery. This requires that all information contained in a description of the electrical properties of the constituents of the medium be retained intact through the various transformations, operations, and manipulations to be prescribed. In effect, there will be presented an alternate set of mathematical equations to that of Maxwell through which the electromagnetics problem of which they are a model may be solved. It is claimed that the new set of equations may profitably and without loss be considered as an

alternate starting point for the investigation of a large class of problems in electromagnetic theory.

Beyond that, it will be shown that the new formulation is subject to a systematic process for extracting the complete solution to the problem. Thus, a unified, systematic, formal solution to Maxwell's equations for linear media will be presented. The solution will, of course, be only formal, for the machinery developed must await an input in the form of a precise description of the medium for which the solution is desired before it can yield the dispersion relation or other existence conditions, as well as the normal modes themselves, as the output.

The resolution process to be developed may become exceedingly cumbersome in many cases, as a host of quadratures may be called for. This will be a manifestation of the complex nature of the problems to be attacked and of the generality of the method. Automatic computers of a high degree of sophistication may well be required to render the process an efficient one. However, an added feature of the method is that the complexity of the calculations will be to some extent under the control of the user.

No claim is made to adherence to strict mathematical rigor in what follows. The arguments presented are intended to demonstrate the plausibility of the results. Historical precedents for various methods to be employed will be indicated as the appropriate stages are reached. A computationally trivial illustration of the theory will be given which is, however, sufficiently general to be of interest. The example will be further specialized to a specific medium of the gyrotropic type in order to illustrate the nature of the computations required. Finally,

possible refinements of the theory will be indicated, with suggestions for improving and extending the validity, facility, applicability, and utility of this reformulation of Maxwell's equations.

## 2. SCOPE OF PROBLEM

The problem to be attacked herein is that of determining the characteristics of the electromagnetic fields which may exist in a given linear medium in response to known sources. To be more precise, the restrictions on the medium are that there be a definite, linear, time-invariant relation among the field intensity, flux density, and current density vectors at each point in space. These conditions are sufficiently weak to permit consideration of a medium which is nonuniform, inhomogeneous, anisotropic, lossy, with or without sources. It is required merely that the region be describable electrically by giving the three tensor fields of capacitivity, permeability, and conductivity at every point. The problem is to extract from this data the characteristics of the macroscopic electromagnetic fields which may be excited in the region of interest by sources within or without it.

The time-invariance imposed upon the medium permits a considerable simplification of the problem in that a harmonic, steady-state analysis will suffice, with little loss of generality. In addition, this condition permits the combination of two of the three tensor fields, the capacitivity and conductivity, into a single permittivity tensor field. Thus, an arbitrary geometrical configuration of lossy dielectric and permeable materials with any physical, even discontinuous, variation of electrical properties is describable by two dyadic functions of position and frequency. The condition that the relation between the field intensities and flux densities be an unambiguous one imposes one final restriction upon the systems to be considered. It is required that the

constitutive dyadics, both the permittivity and the permeability, have nonzero eigenvalues.

In a system satisfying the conditions outlined above, all the relevant electrical information concerning the system will be contained in the two constitutive tensor fields. They alone are sufficient to prescribe the form of the normal modes of the medium. If the sources, both the electric and magnetic current densities, be specified as well, the two dyadics will determine the actual electromagnetic field configuration everywhere in space. It is proposed to prescribe a set of operations for transforming the information contained in the two constitutive dyadic functions into information about the electromagnetic fields which will exist in the given medium.

Under the stipulated conditions, the macroscopic electromagnetic fields and source currents will be complex vector functions of position, time-harmonic at the radian frequency  $\omega$ , and describable by three-component vector functions in some arbitrarily chosen coordinate system. The electrical constitution of the medium is specified by three tensor fields: the relative capacitivity,  $K(r)$ , the relative permeability,  $\mu(r)$ , and the conductivity,  $\sigma(r)$ , where  $r$  represents the position vector. By virtue of the nonuniformity and inhomogeneity of the medium, these tensors are functions of position which may have discontinuities, for example at the interface between a dielectric and free space. If the medium is isotropic, the tensors will reduce to scalars.

In order to exhibit Maxwell's equations explicitly, the following definitions will be used. They are appropriate for the rational mks system of units.

Permeability of free space:	$\mu_0$	
Capacitivity of free space:	$\epsilon_0$	
Speed of light in vacuo:	$c$	$c^2 \mu_0 \epsilon_0 = 1$
Intrinsic impedance in vacuo:	$\eta$	$\eta^2 = \mu_0 / \epsilon_0$
Wave number in vacuo:	$k$	$k = \omega / c$
Imaginary unit:	$j$	$j^2 = -1$

The permittivity tensor field is a combination of the relative capacity and conductivity tensors:

$$\epsilon(r) = K(r) - j(\eta/k)\sigma(r) \quad (2.1)$$

The permittivity and permeability dyadics,  $\epsilon(r)$  and  $\mu(r)$ , have been assumed to be nonsingular.

For convenience, the electromagnetic field quantities will be expressed in the following form, wherein  $E(r)$ ,  $M(r)$ ,  $C_e(r)$ , and  $C_m(r)$  are complex, vector functions of position and, implicitly, of the wave number.

$$\text{Electric field} \quad E(r)e^{j\omega t} \quad (2.2)$$

$$\text{Magnetic field} \quad (j/\eta)M(r)e^{j\omega t} \quad (2.3)$$

$$\text{Electric current density} \quad (k/j\eta)\epsilon(r)C_e(r)e^{j\omega t} \quad (2.4)$$

$$\text{Magnetic current density} \quad k\mu(r)C_m(r)e^{j\omega t} \quad (2.5)$$

In terms of these quantities, Maxwell's equations take the form

$$\text{curl } E = k\mu M - k\mu C_m \quad (2.6)$$

$$\text{curl } M = k\epsilon E - k\epsilon C_e \quad (2.7)$$

These equations are to be satisfied at every point of the space. Boundary conditions, other than the regularity of the solutions, need not be



specified since they have been incorporated in the spatial variation of the constitutive tensors,  $\epsilon(r)$  and  $\mu(r)$ .

### 3. OPERATOR FORMULATION

Maxwell's equations are a pair of coupled vector partial differential equations with variable tensor coefficients. For the purpose of exhibiting a general solution to these equations, it is convenient to reformulate them as a single operator equation. This eclipses the multiple and coupled character of the equations and permits consideration of Maxwell's equations as an abstract relation between the sources and the resultant fields. The formulation will be that of an abstract operator acting upon an abstract quantity representing the response to transform it into one that represents the excitation. Ultimately, this will facilitate the concretization of an abstract solution to this general problem.

Operator methods are a well-known and potent tool in quantum mechanics, but less common in electromagnetic theory. As those of a linear algebra, such methods have an intrinsic value in educing certain invariant properties of the solutions to operator equations, such as reciprocity, or the reality of the eigenvalues. As such, these operator methods have been explored by Bresler, Marcuvitz, and others<sup>1-8</sup> for their applicability to electromagnetic theory problems. Herein, these aspects of the operator calculus will not be pursued; it will be introduced mainly to facilitate the formulation of an abstract, formal solution to Maxwell's equations.

From the operator point of view, Maxwell's equations prescribe a set of operations to be performed upon the electromagnetic field and source vectors, the results of which, when equated, express the physical

interrelation among the fields. As the fields are three-vectors, the operators acting upon them should be, apart from scalar multipliers, three-dyadics. The permittivity and permeability tensors are such dyadics; when applied to the field intensity vectors, they produce the flux density vectors. Similarly, insofar as it produces a new vector field from a given one, the curl operation should be expressible as a dyadic operator.

This is indeed possible, in any particular coordinate system. Thus, in a rectangular coordinate system,  $(x,y,z)$ , the three-dyadic representation of the curl operator is

$$\text{curl} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \quad (3.1)$$

In a cylindrical,  $(\rho, \phi, z)$ , or spherical,  $(r, \theta, \phi)$ , coordinate system, similar dyadic representations are

Cylindrical:

$$\text{curl} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{1}{\rho} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial \rho} \\ -\frac{1}{\rho} \frac{\partial}{\partial \phi} & \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho & 0 \end{bmatrix} \quad (3.2)$$

Spherical:

$$\text{curl} = \begin{bmatrix} 0 & -\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} & \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \sin \theta \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} & 0 & -\frac{1}{r} \frac{\partial}{\partial r} r \\ -\frac{1}{r} \frac{\partial}{\partial \theta} & \frac{1}{r} \frac{\partial}{\partial r} r & 0 \end{bmatrix} \quad (3.3)$$

The curl operator may thus be thought of as an abstract dyadic operator which may be given an explicit expression in any coordinate system.

There may now be introduced two immittance operators as combinations of the above operators, obtained formally by the rules of matrix inversion and multiplication. The impedance operator is a three-dyadic defined as

$$\text{imp} = (k\epsilon)^{-1} \text{curl} \quad (3.4)$$

and the admittance operator is

$$\text{adm} = (k\mu)^{-1} \text{curl} \quad (3.5)$$

The two field intensity vectors may be combined into a single, abstract, six-component field vector,  $S(r)$ , and the two excitation vectors into a source six-vector,  $C(r)$ , whose structures are exhibited in the following partitioned column vectors.

$$S(r) = \begin{bmatrix} E(r) \\ M(r) \end{bmatrix} \quad C(r) = \begin{bmatrix} C_e(r) \\ C_m(r) \end{bmatrix} \quad (3.6)$$

The Maxwell operator is an abstract six-dyadic operator which will operate upon such six-vectors. Its structure is given by the following partitioned matrix.

$$\text{maxl} = \begin{bmatrix} 1 & -\text{imp} \\ -\text{adm} & 1 \end{bmatrix} \quad (3.7)$$

Here, 1 represents the unit three-dyadic.

In terms of the above abstract quantities, Maxwell's equations become the single operator equation

$$\text{maxl} \, S(r) = C(r) \quad (3.8)$$

The six-dyadic, differential operator, maxl, incorporates the physical interaction among the electromagnetic fields as well as the electrical properties of the medium.

In abstract terms, the solution to the problem requires the determination of the operator inverse to maxl, that is, that operator or process which, when applied to the source, C(r), will yield the fields, S(r).

#### 4. TRANSFORM FORMULATION

The operator form of Maxwell's equations, Eq. (3.8), involves a differential operator acting upon the unknown function. A direct integration of this differential equation, say by substitution of trial functions, is not feasible since the solution cannot be expected to be expressible in closed form in terms of known, tabulated, elementary functions. Realization of this raises the question of just what will constitute a solution to the problem; that is, how may the six-component vector function of position,  $S(r)$ , which solves the equation be exhibited?

This question of the presentation of the solution may be resolved by choosing some convenient basis of representation for the function to be described, much as a coordinate system is chosen for the mensuration of the space. The basis will consist of a set of functions with the property of completeness, at least with respect to the class of functions which are sufficiently regular to be possible solutions to the physical problem here considered. The unknown solution will then be expressed as an expansion in this complete set. After the arbitrary complete set has been selected, the problem remaining is that of determining the generalized Fourier coefficients of the expansion.

By the above procedure, the problem will have been transformed from the solving of a differential operator equation in position space to the solving of a corresponding equation in the space of the Fourier coefficients. The spatial variables will have been eliminated, just as the Laplace transform typically eliminates the time variable from an

equation by transforming to the frequency domain. Most important, the differential operator in physical space will have been replaced by one of an algebraic or integral nature in transform space.

In electromagnetic theory, a form of the process contemplated here was developed by Schelkunoff<sup>9</sup> to produce the generalized telegraphist's equations. In a typical problem of propagation in a nonuniform waveguide, expansion functions are chosen from the solution of a related, simpler problem, say that of a limiting case of the actual one. The resulting transformation eliminates the transverse variables, so that a set of coupled ordinary differential equations in only the axial variable is obtained for the coefficients of the expansion. These telegraphist's equations are of the nature of transmission line equations and the coupling coefficients provide much information about the propagation of modes in the nonuniform system. This process is somewhat generalized herein, where the expansion functions remain arbitrary, all the spatial variables are eliminated, the differential character of the equation is entirely suppressed, and vector and dyadic functions are treated. The procedure will yield an equation for the transform variables to which a definite set of operations may be applied to extract the complete solution.

The translation from the original to the transform space must be invertible; that is, a solution obtained in transform space should be subject to being carried back to the original domain, if desired. There will then be no loss of information attendant to the consideration of the problem in the transform space.

The original space is indexed by the position vector  $\mathbf{r}$ , which

may be considered a composite index comprising the three coordinates of the space. In the transformation, each coordinate or spatial index is eliminated at the cost of introducing a corresponding transform index, just as the frequency variable supplants the time variable when a Fourier transform is applied to a problem in the time domain. The transform space will thus be indexed by a composite index,  $s$ . The transformation of a function of position,  $g(r)$ , is achieved by multiplying it by the transformer  $t(s,r)$  and integrating throughout the space. This eliminates the position index  $r$  and leaves the transform  $G(s)$ , a function of the composite index  $s$ . The inverse transformation is achieved by multiplying the transform  $G(s)$  by the transformer  $T(r,s)$  and integrating or summing, as appropriate, over each variable of the composite index,  $s$ , leaving the function of position,  $g(r)$ .

Such multiple integrations and summations as are involved in the above processes will here be denoted by the generic summation symbol  $\$$ . To be precise, this symbol will dictate that the expression which follows it be integrated or summed over the complete range of the variables which are contained in the repeated, dummy, composite indices appearing in the summand. Summation will be implied for the discrete variables in the composite index and integration, possibly with appropriate weight functions, for the continuous variables. Thus, the above transform pair will be written

$$G(s) = \$ t(s,r)g(r) \qquad g(r) = \$ T(r,s)G(s) \qquad (4.1)$$

The unit function for such summations will be denoted by  $1(u,s)$ . It has the property expressed by

$$G(u) = \$ 1(u,s)G(s) \qquad (4.2)$$



This idemfactor thus comprises a Dirac delta function for each variable in the composite index whose range is a continuum, a Kronecker delta for each discrete variable, and a unit dyadic, as required.

As an illustration of the use of this compact notation, consider a function of position,  $g(r)$ , in a cylindrical coordinate system to which it is desired to apply a Fourier-Bessel transform. In this case, the composite index  $r$  represents  $(\rho, \phi, z)$  and the transformation will lead to a space indexed by  $s = (q, n, \beta)$ . The transformer  $t(s, r)$  and its inverse  $T(r, s)$  are given by

$$t(s, r) = t(q, n, \beta; \rho, \phi, z) = e^{-j(n\phi + \beta z)} J_n(q\rho) \quad (4.3)$$

$$T(r, s) = T(\rho, \phi, z; q, n, \beta) = (2\pi)^{-2} e^{j(n\phi + \beta z)} J_n(q\rho) \quad (4.4)$$

The ranges of  $\rho$  and  $q$  are 0 to  $\infty$ , with weight functions  $\rho$  and  $q$ , respectively; the range of  $\phi$  is 0 to  $2\pi$ , that of  $n$  is all positive and negative integers; the ranges of  $z$  and  $\beta$  are  $-\infty$  to  $\infty$ . Accordingly, Eqs. (4.1) would in this case be interpreted as

$$\begin{aligned} G(q, n, \beta) &= \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \int_0^{\infty} \rho d\rho t(q, n, \beta; \rho, \phi, z) g(\rho, \phi, z) \\ g(\rho, \phi, z) &= \int_{-\infty}^{\infty} d\beta \sum_{n=-\infty}^{\infty} \int_0^{\infty} q dq T(\rho, \phi, z; q, n, \beta) G(q, n, \beta) \end{aligned} \quad (4.5)$$

For the problem at hand, the spatial functions and their transforms are six-vectors. Correspondingly, the transformers will be six-dyadics. In fact, let there be chosen some convenient dyadic complete orthonormal set  $c(r, s)$  with its inverse  $d(s, r)$ . With no loss of generality, these six-dyadics may be taken to be diagonal. The completeness property is expressed by

$$\$ c(r,s)d(s,p) = l(r,p) \quad (4.6)$$

and that of orthonormality by

$$\$ d(u,r)c(r,s) = l(u,s) \quad (4.7)$$

The transformation law for a column vector  $v(r)$  and the inversion of its transform  $V(s)$  will then be

$$v(r) = \$ c(r,s)V(s) \quad V(s) = \$ d(s,r)v(r) \quad (4.8)$$

The transformation of a row vector  $w(r)$  is to be performed in the converse manner.

$$w(r) = \$ W(s)d(s,r) \quad W(s) = \$ w(r)c(r,s) \quad (4.9)$$

The properties expressed by Eqs. (4.6) and (4.7) then guarantee that the form of the scalar inner product of two vectors will be preserved by the transformation.

$$\$ w(r)v(r) = \$ W(s)V(s) \quad (4.10)$$

Another consequence of these transformation laws is that a dyadic kernel operator equation in position space such as  $x(r) = \$ A(r,p)v(p)$  will be translated into an equation of similar form in transform space,  $X(u) = \$ B(u,s)V(s)$ , provided that the transformation law for the dyadic operator  $A(r,p)$  is taken as

$$B(u,s) = \$ d(u,r)A(r,p)c(p,s) \quad (4.11)$$

It follows that any linear dyadic operator in position space whose nature is such that there could be constructed for it a representation as a kernel of an integral operator will have a representation in transform space in the form of a dyadic function of two composite transform indices. This transform of the operator may be obtained by allowing the position-space operator to act upon  $c(r,s)$ , premultiplying the

result by the dyadic  $d(u,r)$  and integrating throughout the space, as in Eq. (4.11). The actual kernel representation of the operator in coordinate space need not be found explicitly; indeed, that kernel might be highly singular, possibly involving delta functions and their derivatives. This situation is analogous to that which obtains in the quantum mechanical representation of dynamical variables as operators.<sup>10-13</sup>

The hypothesis is now made that the abstract  $\text{maxl}$  operator of Eq. (3.8) is such an operator. The rigorous justification of this hypothesis will be held in abeyance while its consequences are explored.

Proceeding now with the transformation of the operator form of Maxwell's equations, let  $F(s)$  and  $Q(s)$  be the transforms of the vectors  $S(r)$  and  $C(r)$ , respectively. That is,

$$S(r) = \int c(r,s)F(s) \quad F(s) = \int d(s,r)S(r) \quad (4.12)$$

$$C(r) = \int c(r,s)Q(s) \quad Q(s) = \int d(s,r)C(r) \quad (4.13)$$

In accordance with the preceding discussion, let the transform of the  $\text{maxl}$  operator be  $D(u,s)$ .

$$D(u,s) = \int d(u,r) \text{maxl } c(r,s) \quad (4.14)$$

Upon premultiplying both sides of Eq. (3.8) by  $d(u,r)$ , substituting the expansion for  $S(r)$ , and integrating, Maxwell's equations become the operator transform equation

$$\int D(u,s)F(s) = Q(u) \quad (4.15)$$

## 5. SIGNIFICANCE OF REFORMULATION

The equation obtained here for the transforms of the electromagnetic fields, Eq. (4.15),

$$\oint D(u,s)F(s) = Q(u)$$

merits closer examination. Despite the notational disguise, it is entirely equivalent to the original Maxwell equations.

In this equation,  $F(s)$  is an unknown six-vector, the transform of the desired field vector  $S(r)$ . The six-vector  $Q(s)$  is the known transform of the given source vector  $C(r)$ .  $D(u,s)$  is a six-dyadic kernel, a function of two composite indices of the transform space and, implicitly, of the wave number,  $k$ , as a parameter. It incorporates in an intimate combination

- 1) the physical law of interaction of electromagnetic fields, namely that, in the steady state, one field determines the spatial rate of change of the other, as expressed by the curl operation;
- 2) the electrical constitution of the medium filling the space, as described by the tensor capacitivity, permeability, and conductivity at every point;
- 3) the geometrical configuration of the material bodies whose arrangement in space is incorporated in the spatial variation, particularly the discontinuities, of the constitutive tensors;
- 4) the boundary conditions which would have to be appended to the original partial differential equations to prescribe the field discontinuities at interfaces between regions of different electrical properties;

- 5) the basis of representation selected to express the spatial variation of the fields, as embodied in the dyadic transformers  $c(r,s)$  and  $d(s,r)$ .

The summation operation gives the equation the character of a generalized integral equation of the first kind, with  $Q(s)$  as the forcing function,  $F(s)$  as the unknown function, and  $D(u,s)$  as the kernel.

A review of the procedure involved in obtaining the dyadic kernel  $D(u,s)$  will show that no information about the electromagnetic system has been lost in the transition from the original form of Maxwell's equations to the present formulation. An appropriate coordinate system for the space having been chosen, the medium is first described mathematically by giving the two constitutive dyadics,  $\epsilon(r)$  and  $\mu(r)$ , at each point in space. These two constitutive matrices are inverted and then premultiplied into the dyadic representative of the curl operator to form the immittance operators  $imp$  and  $adm$ , as in Eqs. (3.4) and (3.5). These three-dyadic operators are arranged in a six-dyadic as in Eq. (3.7) to create the  $maxl$  operator for the medium in question. A diagonal dyadic complete set  $c(r,s)$  and its inverse  $d(s,r)$  are selected as a basis of representation. The  $maxl$  operator is applied to  $c(r,s)$  and, finally, the integrations prescribed in Eq. (4.14) are performed to produce the kernel  $D(u,s)$ . As each of these steps, in particular the transformation, is reversible, it is clear that all the pertinent information about the system has been retained in the process of obtaining  $D(u,s)$ . The problem may hence be solved just as thoroughly by an attack on the new equation, Eq. (4.15), as on the original Maxwell equations.

The most important characteristic of  $D(u,s)$  as a replacement for the  $maxl$  operator is the absence of spatial coordinates and of

differential operators. A major advantage of the reformulation of Maxwell's equations lies in that as expressed in this integral equation form, approximation methods, variational techniques, perturbation approaches, and abstract operator analyses may be applied to the problem more easily than in the original formulation as coupled, vector, partial differential, variable coefficient equations. In fact, a complete, exact solution to the transformed equation may be extracted through the application of a definite set of operations to the known kernel  $D(u,s)$  and source  $Q(s)$ . Any solution, however obtained, for  $F(s)$  constitutes the expansion coefficients for the field vector  $S(r)$  in the selected basis of representation  $c(r,s)$ . The solution  $F(s)$  may be transformed back to the desired  $S(r)$ , in accordance with Eq. (4.12), to obtain both the electric and magnetic fields at every point in space.

This last step of transformation back to physical space will often be superfluous as much information may be gleaned from the transform itself. For example, the existence of transverse electric or transverse magnetic waves may be deduced from a simple examination of the transform  $F(s)$  for any vanishing components. Multiple solutions at a given frequency are likewise reflected in the multiplicity of the solution for the transform  $F(s)$  at that wave number.

Less trivially, in the case of a sourceless region for which the normal modes are desired, Maxwell's equations in both the original and transformed forms will be homogeneous and solutions will exist only under certain conditions. These conditions will apply just as well for the solution of the homogeneous equation in transform space. Typically, these existence conditions determine the eigenfrequencies of the system by imposing a restriction on the wave number,  $k$ . Thus if the system

under consideration is a cavity, its resonant frequencies will be obtained in solving the transformed equations, without the necessity of transforming back to coordinate space. Similarly, if propagation of normal modes on a guiding structure is considered, the conditions for the existence of solutions to the homogeneous transformed equation will constitute the dispersion relation for the medium. In a propagation problem the propagation constant appears simply as another parameter and the characteristic existence condition will prescribe a relation between the wave number and the propagation constant.

Still more information about the system may be extracted from the transformed solution by a limited form of inversion of the transforms. Thus, certain quadratic forms in the fields may be evaluated in terms of the transforms without first obtaining the fields explicitly.

Some quadratic forms of importance are

$$W_e(r) = E^+(r)\epsilon(r)E(r) \quad (5.1)$$

$$W_m(r) = M^+(r)\mu(r)M(r) \quad (5.2)$$

$$N(r) = E^*(r) \times M(r) \quad (5.3)$$

where \* indicates the complex conjugate and + the Hermitian conjugate of a quantity. The interest of these quantities arises from the relations

$$\text{time-average complex electric energy density} = (\epsilon_0/4)W_e$$

$$\text{time-average complex magnetic energy density} = (\epsilon_0/4)W_m$$

$$\text{time-average complex Poynting vector} = (1/2j\eta)N^*$$

In order to calculate these expressions with only a partial inversion of the transforms back to position space, some of the dyadics to be used will be partitioned, in the manner of Eqs. (3.6), as follows.

$F(s)$ , the six-vector transform of the field vector  $S(r)$ , partitions as

$$F(s) = \begin{bmatrix} V(s) \\ I(s) \end{bmatrix} \quad (5.4)$$

The diagonal six-dyadic  $c(r,s)$  may be partitioned as

$$c(r,s) = \begin{bmatrix} e(r,s) & 0 \\ 0 & h(r,s) \end{bmatrix} \quad (5.5)$$

The transformation laws for the individual field three-vectors are therefore

$$E(r) = \int e(r,s) V(s) \quad M(r) = \int h(r,s) I(s) \quad (5.6)$$

In addition, let  $\hat{n}$  denote the skew-symmetric dyadic associated with a vector  $n$ . That is,

$$n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \hat{n} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \quad (5.7)$$

The component of the vector  $N(r)$  along any vector  $n(r)$  may then be expressed as

$$N(r) \cdot n(r) = E^+(r) [-\hat{n}(r)] M(r) \quad (5.8)$$

Introducing the three auxiliary three-dyadic functions of three composite indices

$$W^e(u,r,s) = e^+(r,u) e(r) e(r,s) \quad (5.9)$$

$$W^h(u,r,s) = h^+(r,u) h(r) h(r,s) \quad (5.10)$$

$$P_n(u,r,s) = e^+(r,u) [-\hat{n}(r)] h(r,s) \quad (5.11)$$



the three quadratic forms of interest will be obtainable directly from the transforms of the fields through

$$W_e(r) = \int V^+(u) W^e(u, r, s) V(s) \quad (5.12)$$

$$W_m(r) = \int I^+(u) W^m(u, r, s) I(s) \quad (5.13)$$

$$N(r) \cdot n(r) = \int V^+(u) P_n(u, r, s) I(s) \quad (5.14)$$

It may be noted from Eqs. (5.12) and (5.13) that a passive medium will be characterized by dyadics  $W^e$  and  $W^m$  that are positive definite and that this property is directly related to the positive definiteness of the constitutive dyadics, as may be seen from Eqs. (5.9) and (5.10).

Less trivial qualitative and semi-quantitative information about the system will be obtainable from an examination of the character of the Poynting vector field. This may be inferred from Eq. (5.14), in which  $n$  will typically be a unit vector in some direction of interest. Thus, a Poynting vector field which is found to have only an axial component will evidently characterize a propagating wave. Similar qualitative characterizations may be made of such phenomena as reflection, scattering, radiation, backward waves.

Thus, the transform equation, Eq. (4.15), is, in principle, entirely equivalent in content to the original Maxwell equations. It has the merit that its solution may be systematized.

## 6. INTEGRAL EQUATION FORMULATION

As pointed out earlier, the operator transform equation for the electromagnetic fields obtained herein has the aspect of a somewhat generalized integral equation of the first kind. Such equations are generally intractable by their very nature. Furthermore, the quantities involved in the equation are dyadics of sixth order. In effect, then, the transform equation in its present form, Eq. (4.15), may fairly be said to be unmanageable.

The six-dyadic kernel  $D(u,s)$  has, however, been so constructed that a considerable simplification of the equation is possible. The order of the dyadics to be manipulated may firstly be reduced from six to three, at the cost of introducing a pair of coupled equations for the individual field transforms. But then these equations may be recombined into one equation, still with third-order dyadics, of the form of an integral equation of the second kind, which is amenable to analysis. There will then remain merely to adapt the classical Fredholm theory for such equations to the somewhat generalized type of integral equation to be obtained here. This process will lead ultimately to a complete, formal solution to Maxwell's equations.

The reformulation of the equations continues now with the further dissection of the six-dyadics into partitioned matrices. To accompany the partition of the field vector  $F(s)$  in Eq. (5.4), the source vector  $Q(s)$  partitions as

$$Q(s) = \begin{bmatrix} W(s) \\ J(s) \end{bmatrix} \quad (6.1)$$

In view of the construction of the  $\text{maxl}$  operator, Eq. (3.7), and of the orthonormality of the transformers,  $c(r,s)$  and  $d(s,r)$ , as expressed by Eq. (4.7), the six-dyadic kernel  $D(u,s)$  has the following highly significant structure.

$$D(u,s) = \begin{bmatrix} 1(u,s) & -Z(u,s) \\ -Y(u,s) & 1(u,s) \end{bmatrix} \quad (6.2)$$

The three-dyadic immittance kernels,  $Z(u,s)$  and  $Y(u,s)$ , are defined by this equation.

Introducing these partitioned forms of the matrices into the transform equation, Eq. (4.15), it is seen that the equation can be decomposed into the network equations

$$V(u) = W(u) + \int Z(u,s)I(s) \quad (6.3)$$

$$I(u) = J(u) + \int Y(u,s)V(s) \quad (6.4)$$

In these coupled equations, the transformed field three-vectors  $V(s)$  and  $I(s)$  are the unknowns; the three-vectors  $W(s)$  and  $J(s)$  are transforms of given sources; the impedance and admittance kernels  $Z(u,s)$  and  $Y(u,s)$  are known three-dyadics, obtained by inspection of Eq. (6.2). A comparison is invited of these equations with the static network equations of Marcuvitz<sup>14</sup> or with the dynamic telegraphist's equations of Schelkunoff.<sup>9</sup>

The final phase of the reformulation process will now be expounded, with a view to uncoupling the two equations and producing an equation much like an integral equation of the second kind, for which a

general theory may be formulated. The uncoupling process is analogous to that which, for a homogeneous medium, produces the wave equation from the coupled Maxwell equations; it involves simply the substitution of Eq. (6.4) into Eq. (6.3).

Some preliminary definitions will be found to be convenient. It is profitable to introduce a new parameter into the equations, for ease of analysis. While this may be done entirely arbitrarily, it is particularly convenient, in case the constitutive tensors are independent of frequency, to let this new parameter,  $\lambda$ , absorb and fill the role of the wave number,  $k$ . This will facilitate the physical interpretation of some of the equations to be obtained. It should be emphasized that the present restriction to frequency-independent media is made merely for convenience of interpretation and will be lifted later.

The system being studied possesses some characteristic dimensions. From these, let there be selected a convenient quantity,  $A$ , with the dimensions of an area, to characterize the physical size of the system. The dimensionless parameter,  $\lambda$ , is defined as

$$\lambda = 1/k^2 A \quad (6.5)$$

The three-dyadic kernel,  $K(u,s)$ , is defined by

$$K(u,s) = k^2 A \oint Z(u,v)Y(v,s) \quad (6.6)$$

A review of the steps involved in calculating the two impedance kernels  $Z(u,s)$  and  $Y(u,s)$  will readily show that, in the case of a frequency-independent medium, the kernel  $K(u,s)$  is independent of the wave number  $k$ , and hence of the parameter  $\lambda$ . Finally, the compound source,  $U(s)$ , is given by

$$U(u) = W(u) + \oint Z(u,s)J(s) \quad (6.7)$$

A direct substitution of Eq. (6.4) into Eq. (6.3) now yields the operator transform reformulation of the electromagnetics problem in generalized integral equation form,

$$V(u) = U(u) + \lambda \oint K(u,s)V(s) \quad (6.8)$$

This equation is to be solved for the unknown three-vector  $V(s)$  for a given compound source  $U(s)$ , kernel  $K(u,s)$  and parameter  $\lambda$ . When the solution is introduced into Eq. (6.4) to calculate  $I(s)$ , the complete solution for the transformed fields is obtained.

## 7. METHOD OF SOLUTION

The final reformulation of Maxwell's equations obtained here, Eq. (6.8),

$$V(u) = U(u) + \lambda \int K(u,s)V(s)$$

has the aspect of a linear Fredholm integral equation of the second kind. The classical Fredholm theory for such equations must, however, be modified for the case at hand since the equation is somewhat more general than the prototype equation treated by Fredholm. In the present case, a generalized, multiple summation replaces the single integration of the original Fredholm equation, the range of summation is generally not a simple closed interval of integration, and dyadic quantities appear in the summands.

This last innovation, represented by the dyadic character of the kernel, requires a nontrivial modification of the Fredholm theory. For this theory<sup>15-20</sup> is usually developed in terms of determinants with elements formed from the values of the kernel at various points. Were this process carried out formally in the present case, there would be obtained determinants with noncommuting, matrix elements, whereupon the theory would break down. A formulation is required which avoids the formation of determinants and is applicable to this matrix equation.

Such a formulation can be developed, in close analogy to that presented by Smithies,<sup>17</sup> but with certain modifications demanded by the dyadic character of the kernel. The quantities normally given by determinants will, in this matrix case, be expressed in terms of a set of recursion relations.

The method of solution to be presented here takes full advantage of the linearity of the equation by providing a solution for the case of just a unit source function. The solution for an arbitrary source can then be obtained by superposition. The resolution of the problem posed by Eq. (6.8) reduces then to the search for the resolvent kernel  $H(u,s;\lambda)$ , which plays the role of a Green's function for the equation, in terms of which the solution to the equation

$$V(u) = U(u) + \lambda \oint K(u,s)V(s) \quad (7.1)$$

for an arbitrary source  $U(s)$  will be

$$V(u) = U(u) + \lambda \oint H(u,s;\lambda)U(s) \quad (7.2)$$

Upon substituting this assumed solution into Eq. (7.1), noting that it is to be satisfied for all sources  $U(s)$ , it is found that  $H(u,s;\lambda)$  must satisfy the resolvent equation

$$H(u,s;\lambda) = K(u,s) + \lambda \oint K(u,v)H(v,s;\lambda) \quad (7.3)$$

The resolvent  $H(u,s;\lambda)$  is a three-dyadic kernel associated with, and entirely determined by, the kernel  $K(u,s)$  of the equation. A study of its properties is equivalent to an analysis of the original electromagnetics problem. In fact, solving Eq. (7.3) for  $H(u,s;\lambda)$  will yield the solutions for the transformed fields successively from Eqs. (7.2) and (6.4). Some of the more significant properties of the resolvent are expressed by the following relations, which are fairly readily derivable from Eq. (7.3).

$$\oint K(u,v)H(v,s;\lambda) = \oint H(u,v;\lambda)K(v,s) \quad (7.4)$$

$$H(u,s;0) = K(u,s) \quad (7.5)$$

$$\frac{\partial H(u,s;\lambda)}{\partial \lambda} = \oint H(u,v;\lambda)H(v,s;\lambda) \quad (7.6)$$

$$H(u, s; \lambda) - H(u, s; \xi) = (\lambda - \xi) \int H(u, v; \lambda) H(v, s; \xi) \quad (7.7)$$

In Eq. (7.6), no summation over  $\lambda$  is intended, of course, although it is repeated in the summand; it is merely a parameter, not a dummy composite index.

Another important property of the resolvent, following immediately from Eq. (7.2), is that as long as  $H(u, s; \lambda)$  is finite, the homogeneous equation

$$V(u) = \lambda \int K(u, s) V(s) \quad (7.8)$$

will have only the trivial solution  $V(s) = 0$ . The homogeneous equation will have nontrivial solutions, however, but only for certain characteristic values of  $\lambda$ . At these values,  $H(u, s; \lambda)$  will have to become infinite.

In order to gain greater insight into the nature, structure, and properties of the resolvent, it is instructive to consider a very special but important case of Eq. (7.1). This is that of a degenerate kernel, in which case it is possible to give an explicit expression, in closed form, for the resolvent. A degenerate kernel is one that can be factored so as to separate its two composite indices. Explicitly, an  $n \times n$  dyadic kernel  $K(u, s)$  is degenerate if it can be expressed in factored form as

$$K(u, s) = A(u)B(s) \quad (7.9)$$

where  $A(s)$  is an  $n \times r$  dyadic function of just one composite index and  $B(s)$  is an  $r \times n$  dyadic function of one index. Let the obverse of the degenerate kernel be defined as the  $r \times r$  constant dyadic

$$R = \int B(s)A(s) \quad (7.10)$$



In the degenerate case, then, Eq. (7.1) becomes

$$V(u) = U(u) + \lambda A(u) \int B(s)V(s) \quad (7.11)$$

from which there may be obtained a simple matrix equation for the auxiliary unknown  $X = \int B(s)V(s)$  by premultiplying by  $B(u)$  and summing:

$$X = \int B(u)U(u) + \lambda R X \quad (7.12)$$

This equation may immediately be solved for  $X$ , which may in turn be substituted in Eq. (7.11) to obtain the solution for  $V(s)$ . Upon comparing the result with Eq. (7.2), it is seen that the resolvent for this degenerate case is given by

$$H(u,s;\lambda) = A(u)(1 - \lambda R)^{-1} B(s) \quad (7.13)$$

The structure of this explicit expression for the resolvent for a degenerate kernel clarifies many of the properties of resolvents for the general case. The properties stated in Eqs. (7.4) - (7.7) may be readily verified for this resolvent. Of greater importance for the sequel is the fact that the resolvent contains as a common denominator the determinant  $\det(1 - \lambda R)$ . This scalar function of the parameter  $\lambda$  determines the poles of the resolvent considered as a function of  $\lambda$ . The secular equation is

$$\det(1 - \lambda R) = 0 \quad (7.14)$$

and its roots are those values of  $\lambda$  for which the homogeneous equation, Eq. (7.8), has nontrivial solutions. In fact, it may be seen from Eqs. (7.12) and (7.11) that the solutions to the homogeneous equation will be

$$V(s) = A(s) X \quad (7.15)$$

where  $X$  is any eigenvector of the obverse matrix,  $R$ .

The secular equation, Eq. (7.14), is a polynomial equation of

degree  $r$ , so that there will be  $r$  characteristic values of  $\lambda$ , counting multiplicities. This assumes that the order,  $r$ , of the obverse matrix is also its rank; if  $R$  is singular, there will be fewer roots, but then a different factorization of the kernel could have been found for which the obverse matrix would have had a lower order.

These results for the degenerate case will be generalized for that of a general, nondegenerate kernel. The central result will be, as may already be anticipated, that the characteristic values of  $\lambda$  will be given by the roots, not of a polynomial, but of an infinite power series in  $\lambda$ . The explicit expression for the resolvent in the degenerate case given in Eq. (7.13) provides the clue to the structure of the general resolvent.

## 8. STRUCTURE OF RESOLVENT

Eq. (7.13) is a prescription for calculating the resolvent of a degenerate kernel. The poles of the resolvent, which correspond to nontrivial solutions of the homogeneous equation, occur at the roots of the determinant  $\det(1 - \lambda R)$ , a polynomial of degree  $r$  in  $\lambda$ . These characteristic values of  $\lambda$  are an intrinsic, though latent, property of the kernel and the fact that there is a finite number,  $r$ , of such roots, counting multiplicities, is the essential characteristic of a degenerate kernel.

A nondegenerate kernel will inherently possess an infinite number of characteristic values of the parameter  $\lambda$ . No factorization into matrices of finite order as in Eq. (7.9) will then be possible. It is always possible, however, to approximate the kernel arbitrarily closely in such factored form, provided matrices of infinite order are admitted. This is so because this decomposition will then be nothing more than an infinite sum of products of functions of each composite index separately, which certainly can represent any sufficiently regular kernel with any degree of accuracy. The obverse matrix of a nondegenerate kernel so expressed will be of infinite order, the determinant  $\det(1 - \lambda R)$  will be not a polynomial but an infinite power series in  $\lambda$ , and the number of characteristic roots will, consistently, be infinite. The nondegenerate case can thus be considered a limiting case of that of a degenerate kernel, but with the transition of the order,  $r$ , to infinity.

Evidently, however, the calculation of the resolvent of a nondegenerate kernel by Eq. (7.13) with the obverse matrix,  $R$ , and the two

factors,  $A(u)$  and  $B(s)$ , of infinite order is not feasible. Nevertheless, the fact that the resolvent could be so expressed, in a formal sense, provides the clue to the solution of the problem for general kernels. For this possibility of considering a nondegenerate kernel as a limiting case of a degenerate one indicates that the structure of the resolvent will be that of a resolvent for a degenerate kernel and, in fact, any property of a resolvent for a degenerate kernel which does not depend explicitly on the order  $r$  or on its finiteness will be possessed by the resolvent for the general kernel. Recognition of this fact leads to the solution of the problem.

Accordingly, in direct analogy with the degenerate case, the resolvent for a general kernel will be assigned the structure

$$H(u,s;\lambda) = C(u,s;\lambda)/p(\lambda) \quad (8.1)$$

The determinant,  $p(\lambda)$ , is a scalar function of only the parameter  $\lambda$  and is the analog of the determinant  $\det(1 - \lambda R)$  of the degenerate case. The characterizer,  $C(u,s;\lambda)$ , is a three-dyadic function of two composite indices and of  $\lambda$ ; it corresponds to the matrix  $A(u)Q(\lambda)B(s)$  of the degenerate case, where  $Q(\lambda)$  is the adjoint of the matrix  $1 - \lambda R$ .

Both the characterizer and the determinant are unknown. Their relation to the kernel is readily determined from the resolvent equation, Eq. (7.3), by multiplying by  $p(\lambda)$ . The resulting characterizer equation is

$$C(u,s;\lambda) = K(u,s)p(\lambda) + \lambda \int K(u,v)C(v,s;\lambda) \quad (8.2)$$

This is, of course, insufficient to determine both  $C(u,s;\lambda)$  and  $p(\lambda)$ . A further, though still partial, specification of the determinant may be made in analogy with the degenerate case by fixing the arbitrary

multiplicative constants of  $C(u,s;\lambda)$  and  $p(\lambda)$  in accordance with

$$p(0) = 1 \quad (8.3)$$

It follows from Eq. (7.5) that

$$C(u,s;0) = K(u,s) \quad (8.4)$$

Further progress toward the determination of  $C(u,s;\lambda)$  and  $p(\lambda)$  may be made by prescribing the structure of the determinant to be the analog of that of the degenerate case; that is,  $p(\lambda)$  is to be a polynomial in  $\lambda$ , but of infinite degree:

$$p(\lambda) = \sum_{n=0}^{\infty} p_n \lambda^n \quad (8.5)$$

Eq. (8.3) prescribes that  $p_0 = 1$ ; all other coefficients remain unknown. Similarly, the characterizer will be expanded as

$$C(u,s;\lambda) = \sum_{n=0}^{\infty} C_n(u,s) \lambda^n \quad (8.6)$$

Here,  $C_0(u,s) = K(u,s)$  and the other three-dyadic coefficients are unknown. Upon substitution of these assumed expansions into the characterizer equation, Eq. (8.2), there is obtained the recursive characterizer equation

$$C_n(u,s) = K(u,s)p_n + \sum K(u,v)C_{n-1}(v,s) \quad (8.7)$$

Eq. (8.7) is a recursion relation for the characterizer coefficients, but it requires a knowledge of the determinant coefficients,  $p_n$ , for the successive calculation of the  $C_n(u,s)$ , starting from the known initial coefficients

$$C_0(u,s) = K(u,s) \quad p_0 = 1 \quad (8.8)$$

There remains, then, to specify the determinant coefficients, whereupon all the characterizer coefficients will be obtainable in succession from

the recursive characterizer equation, and from these the complete solution for the general kernel. But, at this point, an impasse appears to have been reached since all information about the resolvent has already been utilized in obtaining the characterizer equation and no information at all is available for the specification of the determinant coefficients.

This, however, is clearly as it should be, for two unknown quantities were introduced in Eq. (8.1) to replace the one unknown  $H(u,s;\lambda)$ . The determinant coefficients, exclusive of  $p_0$ , are therefore, in fact, entirely arbitrary. Any chosen set of coefficients  $p_n$  may be used in Eq. (8.7) to obtain the corresponding set of characterizer coefficients  $C_n(u,s)$  and, from these, the resolvent  $H(u,s;\lambda)$ . The resultant expression will, of course, be valid for only that range of the parameter  $\lambda$  for which both series of Eqs. (8.5) and (8.6) are convergent.

In view of the arbitrariness of the determinant coefficients, it is tempting to simplify the expressions by selecting zero as the value of all determinant coefficients, except  $p_0$ . This eliminates the question of the convergence of the series for the determinant and simplifies the recursive characterizer equation to

$$C_n(u,s) = \sum K(u,v)C_{n-1}(v,s) \quad (8.9)$$

which, with  $C_0(u,s) = K(u,s)$ , readily yields each characterizer coefficient in succession. With this choice for the  $p_n$ , the final expression for the resolvent is

$$H(u,s;\lambda) = \sum_{n=0}^{\infty} K^{n+1}(u,s)\lambda^n \quad (8.10)$$

where the "powers" of the kernel are defined recursively by

$$K^1(u,s) = K(u,s) \quad K^{n+1}(u,s) = \sum K(u,v)K^n(v,s) \quad (8.11)$$

This is certainly a proper solution to the resolvent equation. It is, in fact, the Neumann series solution which could have been obtained directly from the resolvent equation, Eq. (7.3), by iteration; i.e., by introducing the entire right-hand side of this equation into the summand appearing therein, and repeating the process indefinitely. This Neumann solution has the drawback, however, that the range of  $\lambda$  for which it converges is too small. The series of Eq. (8.10) actually diverges for all values of  $\lambda$  with absolute value greater than that of the characteristic value of smallest magnitude. As has been remarked, the characteristic values of  $\lambda$  and the associated solutions of the homogeneous equation are of the greatest interest, but the Neumann series fails to converge as soon as the first characteristic value is attained. This solution is thus entirely useless for the study of the sourceless solutions.

This is the crux of the problem. Although the determinant coefficients are arbitrary, for a given assignment of values to the  $p_n$  the series for  $p(\lambda)$  and for  $C(u,s;\lambda)$  will converge only for  $\lambda$  within some finite circle of convergence and the resulting expression for the resolvent will be valid only in some restricted range of the parameter  $\lambda$ . This situation clearly defeats the purpose of expressing the resolvent in terms of a characterizer and determinant. What was intended was that the poles of the resolvent, which correspond to the characteristic values of  $\lambda$  and the sourceless solutions, be obtainable as the roots of the determinant. The secular equation would then be simply

$$p(\lambda) = 0 \quad (8.12)$$

with an infinite number of roots corresponding precisely to the infinite

number of characteristic values of  $\lambda$  associated with a nondegenerate kernel. Furthermore, the condition that Eq. (8.12) specify the characteristic values of  $\lambda$  was to lead directly to the actual nontrivial sourceless solutions as well. Thus, if  $\lambda_0$  solves Eq. (8.12), then from Eq. (8.2)

$$C(u, s; \lambda_0) = \lambda_0 \oint K(u, v) C(v, s; \lambda_0) \quad (8.13)$$

and a solution to the homogeneous equation, Eq. (7.8), is

$$V(u) = C(u, s_0; \lambda_0) c_0 \quad (8.14)$$

where  $c_0$  is an arbitrary, constant three-vector and  $s_0$  is any value of the composite index  $s$ . This sourceless solution will be nontrivial, provided merely that  $s_0$  and  $c_0$  be not chosen so as to annihilate the resulting  $V(u)$ .

To fulfill these desired conditions, a set of values for the determinant coefficients must be so specified that, firstly, both  $p(\lambda)$  and  $C(u, s; \lambda)$  be entire functions of  $\lambda$ , thereby insuring convergence for all  $\lambda$  and the validity of the expression for the resolvent over the full frequency spectrum; secondly, that the secular equation, Eq. (8.12), yield all characteristic values and only the characteristic values of  $\lambda$ ; and thirdly, that the eigensolutions  $C(u, s_0; \lambda_0) c_0$  be obtained as well. Now in the degenerate case, in which  $p(\lambda)$  is  $\det(1 - \lambda R)$ , these conditions are satisfied. To achieve these results in the general case, the determinant must be specified in the same way as in the degenerate case, despite the infinite order of the obverse matrix. That is, the relation which is to be specified so as to determine, in conjunction with Eq. (8.2), the desired form of the determinant must be precisely that which obtains in the degenerate case, independently of the order of the obverse matrix. This relation must be investigated.



## 9. TERMINATION CONDITION

By the foregoing reasoning, the problem has been reduced to that of specifying a relation between the characterizer and the determinant of the resolvent for a general kernel which will yield the optimum set of coefficients for the expansion of the determinant. This optimum condition will be such that the poles of the resolvent will be the roots of the determinant, both the characterizer and determinant being entire functions of the parameter  $\lambda$ . The relation to be found will complement that of Eq. (8.2) and thereby provide a sufficient number of conditions for the unique determination of both the characterizer and the determinant.

In the degenerate case, the determinant and the characterizer are

$$p(\lambda) = \det(1 - \lambda R) \quad C(u, s; \lambda) = A(u)Q(\lambda)B(s) \quad (9.1)$$

where  $Q(\lambda)$  is the adjoint of the matrix  $(1 - \lambda R)$ :

$$Q(\lambda) = (1 - \lambda R)^{-1} \det(1 - \lambda R) \quad (9.2)$$

Clearly, there is an intimate relation between  $p(\lambda)$  and  $C(u, s; \lambda)$ , as a result of which, effectively, both of these are entire functions and the characteristic values are given by the roots of  $p(\lambda)$ , regardless of the order of the obverse matrix,  $R$ . As expressed in Eq. (9.1), however, both  $p(\lambda)$  and  $C(u, s; \lambda)$  depend on  $R$  and can be calculated explicitly only if its order,  $r$ , is finite. In order to adapt the specification of Eq. (9.1) to the case of a general kernel, it is necessary to express the relation between the characterizer and determinant in a form which is

independent of the order,  $r$ , of the degeneracy. Such a relation will be applicable to the case of an obverse matrix of infinite order as well as to the degenerate case. The determinant and characterizer of a degenerate kernel are entire functions because they are polynomials in  $\lambda$ ; in the nondegenerate case their expansions in powers of  $\lambda$  will not terminate, but if the condition on the determinant is taken as the analog of that which holds in the degenerate case no singularities will be introduced into either the characterizer or determinant to limit their radii of convergence.

The relation sought is one between the scalar  $p(\lambda)$  and the matrix function of two composite indices  $C(u,s;\lambda)$ . The composite indices can be readily eliminated by summing over them; there will then remain to extract a scalar from the residual matrix. A significant set of scalars associated with a matrix, closely related to its eigenvalues but more easily calculated, consists of the traces of the matrix. An  $n \times n$  matrix  $A$  is characterized by  $n$  traces. The  $m$ -th order trace,  $\text{Tr}_m A$ , is the sum of the principal minors of order  $m$  of the matrix. Its significance lies in the fact that  $\text{Tr}_m A$  equals the sum of the products of the eigenvalues taken  $m$  at a time. The trace of  $A$ , i.e., the sum of its diagonal elements,  $\text{Tr} A$ , corresponds to  $\text{Tr}_1 A$ ; the determinant  $\det A$  coincides with  $\text{Tr}_n A$ . For convenience, the further definitions will be made that  $\text{Tr}_0 A = 1$  and that  $\text{Tr}_m A = 0$  for any  $m$  which exceeds the order  $n$ .

From the relation between the traces and eigenvalues of a matrix, the following properties of the traces may readily be demonstrated.

$$\text{Tr}_m AB = \text{Tr}_m BA \quad (9.3)$$

$$\text{Tr}_m cA = c^m \text{Tr}_m A \quad (9.4)$$

$$\text{Tr}_m A^{-1} = \text{Tr}_{n-m} A / \det A \quad (9.5)$$

$$\text{Tr}_m (1 + A) = \sum_{k=0}^m \binom{n-k}{n-m} \text{Tr}_k A \quad (9.6)$$

The commutation property of Eq. (9.3) holds even if  $A$  and  $B$  are not individually square matrices, although they must be conformable in both orders. This property results in the invariance of the traces under similarity transformations. In Eq. (9.4),  $c$  is a scalar multiplier.

Virtually the only significant scalar which can be extracted from  $C(u, s; \lambda)$  by linear operations is

$$f(\lambda) = \$ \text{Tr } C(s, s; \lambda) \quad (9.7)$$

It may be expected that this scalar function of  $\lambda$  is related to  $p(\lambda)$ . This relation constitutes the Termination Theorem, which can now be proved, granted the preliminary lemma that the operators  $\$$  and  $\text{Tr}$  commute. There follows

$$\begin{aligned} f(\lambda) &= \$ \text{Tr } A(s)Q(\lambda)B(s) = \$ \text{Tr } Q(\lambda)B(s)A(s) \\ &= \text{Tr } \$ Q(\lambda)B(s)A(s) = \text{Tr } Q(\lambda) R \end{aligned}$$

where the commutation property of Eq. (9.3) and the linearity of  $\$$  have been used. Let

$$G(\lambda) = -\lambda R \quad (9.8)$$

Hence, by Eqs. (9.4), (9.2), and (9.1),

$$-\lambda f(\lambda) = \text{Tr } QG = \text{Tr } (1+G)^{-1} pG = p \text{Tr } (1+G^{-1})^{-1}$$

Using Eq. (9.5) and noting that  $\det(1+G^{-1}) = \det[G^{-1}(1+G)]$

$$= \det(1+G) / \det G = p / \det G,$$

$$-\lambda f(\lambda) = p \text{Tr}_{r-1}(1+G^{-1}) / \det(1+G^{-1}) = \det G \text{Tr}_{r-1}(1+G^{-1})$$

From Eqs. (9.6) and (9.5),

$$\begin{aligned} -\lambda f(\lambda) &= \det G \sum_{k=0}^{r-1} (r-k) \text{Tr}_k G^{-1} \\ &= \det G \sum_{k=0}^{r-1} (r-k) \text{Tr}_{r-k} G / \det G \\ &= \sum_{m=1}^r m \text{Tr}_m G \end{aligned}$$

which, from Eqs. (9.8) and (9.4), is equivalent to

$$f(\lambda) = - \sum_{m=1}^r (-1)^m \text{Tr}_m R m \lambda^{m-1}$$

which may be recognized to be

$$f(\lambda) = - \frac{d}{d\lambda} \sum_{m=0}^r (-1)^m \text{Tr}_m R \lambda^m$$

But, from Eq. (9.6),

$$p(\lambda) = \det(1 - \lambda R) = \sum_{m=0}^r (-1)^m \text{Tr}_m R \lambda^m \quad (9.9)$$

Hence  $f(\lambda) = -dp/d\lambda$  and the relation sought is

$$-\frac{dp}{d\lambda} = \oint \text{Tr } C(s, s; \lambda) \quad (9.10)$$

which depends explicitly on neither  $R$  nor its order,  $r$ .

Conversely, it may be shown that the power series for both the determinant and characterizer will indeed terminate for a degenerate kernel if the condition of Eq. (9.10) is imposed in addition to Eq. (8.2). For in the degenerate case, Eq. (8.2) alone yields as the characterizer, for any  $p(\lambda)$ ,

$$C(u, s; \lambda) = A(u)(1 - \lambda R)^{-1} p(\lambda) B(s) \quad (9.11)$$

so that  $\oint \text{Tr } C(s, s; \lambda) = p(\lambda) \text{Tr } (1 - \lambda R)^{-1} R$ , from which

$$\begin{aligned} -(1/p)(dp/d\lambda) &= - (d/d\lambda) \log p = \text{Tr}(1 - \lambda R)^{-1} R \\ &= - \text{Tr } (d/d\lambda) \log (1 - \lambda R) \end{aligned}$$

or, in view of the commutativity of  $\text{Tr}$  and  $d/d\lambda$  and of the initial conditions on  $p(\lambda)$ ,

$$\log p(\lambda) = \text{Tr} \log (1 - \lambda R) \quad (9.12)$$

But since  $\text{Tr} \log A = \log \det A$ , as follows from the relation between the trace and eigenvalues of the matrix  $\log A$ , Eq. (9.12) implies

$$p(\lambda) = \det (1 - \lambda R) \quad (9.13)$$

so that the power series for  $p(\lambda)$  will terminate at  $\lambda^r$ ; in fact,

$$p(\lambda) = \sum_{m=0}^r (-1)^m \text{Tr}_m R \lambda^m \quad (9.14)$$

The Cayley-Hamilton theorem<sup>21,22</sup> may be written

$$\sum_{m=0}^r (-1)^{r-m} \text{Tr}_{r-m} R R^m = 0 \quad (9.15)$$

or, by comparing the coefficients in Eqs. (9.14) and (9.15),

$$\sum_{m=0}^r p_{r-m} R^m = 0$$

By iterating in Eq. (8.7), there is obtained

$$C_r(u, s) = \sum_{m=0}^r K^{m+1}(u, s) p_{r-m} \quad (9.16)$$

But, from Eq. (8.11),

$$K^{n+1}(u, s) = A(u) R^n B(s) \quad (9.17)$$

so that

$$C_r(u, s) = A(u) \sum_{m=0}^r p_{r-m} R^m B(s) = 0 \quad (9.18)$$

Since  $p_{r+1} = 0$  and  $C_r(u, s) = 0$ , Eq. (8.7) implies that all higher-order coefficients of  $C(u, s; \lambda)$  vanish. Hence both the determinant and characterizer expansions for a degenerate kernel will terminate as a

result of the application of Eqs. (8.2) and (9.10).

If the termination condition is applied to the nondegenerate case, the series will not actually terminate, but its effect, which is to cancel the poles of  $(1 - \lambda R)^{-1}$  with the zeros of  $p(\lambda)$  in Eq. (9.11) so as to render  $C(u, s; \lambda)$  an entire function, will persist even as the order of the degeneracy,  $r$ , is allowed to become infinite. Thus, the termination condition, Eq. (9.10), realizes the proper pole cancellation property of the optimum determinant.

## 10. COMPLETE SOLUTION

The characterizer equation and the termination condition, together with the initial conditions, provide the complete solution to the problem posed herein. These equations are

$$C(u,s;\lambda) = K(u,s)p(\lambda) + \lambda \oint K(u,v)C(v,s;\lambda) \quad (10.1)$$

$$-\frac{dp}{d\lambda} = \oint \text{Tr } C(s,s;\lambda) \quad (10.2)$$

$$p(0) = 1 \quad C(u,s;0) = K(u,s) \quad (10.3)$$

Upon solving these for the characterizer  $C(u,s;\lambda)$  and determinant  $p(\lambda)$ , the resolvent  $H(u,s;\lambda)$  is obtained as their ratio, Eq. (8.1). This immediately yields the field transforms  $V(s)$  and  $I(s)$  from Eqs. (7.2) and (6.4) and these may be inverted to obtain the actual fields in coordinate space.

Perhaps of greater importance than the fields for the study of a given linear medium is the fact that the eigenfrequencies are obtainable from the secular equation

$$p(\lambda) = 0 \quad (10.4)$$

This equation, the central result of the theory, has the significance of a dispersion relation for a propagation problem, or of a resonance condition for a cavity, or of any equivalent relation among the frequency of a wave, its wavelength and other characteristics, and the dimensions and electrical properties of the medium. As such, the secular equation affords a powerful characterization of the electromagnetic properties of the medium.

In conjunction with the solution to the secular equation, there is obtained in this formulation the associated characterizer, which corresponds essentially to the residue of the resolvent at its pole. This characterizer yields the solutions to the homogeneous equation, Eq. (7.8), as

$$V(s) = C(s, s_0; \lambda_0) c_0 \quad (10.5)$$

where  $\lambda_0$  is a root of the secular equation,  $s_0$  is any value of the composite index, and  $c_0$  is an arbitrary constant three-vector. The constant vector  $c_0$ , together with the selected value of  $s_0$ , plays the role of the arbitrary amplitude coefficient for the normal mode.

The actual solution of Eqs. (10.1) and (10.2) may proceed by recursion. The determinant and characterizer are expanded in power series in  $\lambda$ , as in Eqs. (8.5) and (8.6), which, upon substitution into Eqs. (10.1) and (10.2) and comparison of coefficients, yield

$$C_n(u, s) = K(u, s) p_n + \sum K(u, v) C_{n-1}(v, s) \quad (10.6)$$

$$-n p_n = \sum \text{Tr } C_{n-1}(s, s) \quad (10.7)$$

$$p_0 = 1 \quad C_0(u, s) = K(u, s) \quad (10.8)$$

These equations suffice to yield all the unknown coefficients of both  $p(\lambda)$  and  $C(u, s; \lambda)$  in succession. Starting with Eq. (10.8),  $p_1$  is obtainable from Eq. (10.7), whereupon  $C_1(u, s)$  can be calculated from Eq. (10.6) and this, in turn, will yield  $p_2$  through Eq. (10.7); the process may be repeated indefinitely to obtain, in principle, explicit expansions of the determinant and the characterizer. The rapidity of convergence of these expansions depends, of course, on the original choice



of expansion functions,  $c(r,s)$ , for the transformation. The solution of the problem would then proceed with the solution and study of the secular equation, Eq. (10.4).

There can arise a situation in which the procedure just described may be thwarted at its start. This is the case if  $K(u,s)$  is infinite or undefined for  $u = s$ , whereupon the prescription for calculating  $p_1$  from Eq. (10.7) is ineffectual. This situation demands a modification of the recursion relations which will sidestep the difficulty.

That such a modification is possible is the result of the residual ambiguity of the characterizer and determinator. The essential quantity is the resolvent and this has been found as the ratio  $H(u,s;\lambda) = C(u,s;\lambda)/p(\lambda)$ . Both numerator and denominator of this ratio may be multiplied by some scalar function  $g(\lambda)$  to yield the equivalent ratio

$$H(u,s;\lambda) = D(u,s;\lambda)/q(\lambda) \quad (10.9)$$

where

$$q(\lambda) = p(\lambda)g(\lambda) \quad D(u,s;\lambda) = C(u,s;\lambda)g(\lambda) \quad (10.10)$$

To preserve the initial value of the determinator,  $g(0) = 1$  may be specified. The modified characterizer equation is then essentially the same,

$$D(u,s;\lambda) = K(u,s)q(\lambda) + \lambda \sum K(u,v)D(v,s;\lambda) \quad (10.11)$$

but the termination condition is altered to

$$-\frac{dq}{d\lambda} = \sum \text{Tr } D(s,s;\lambda) - t(\lambda)q(\lambda) \quad (10.12)$$

where

$$t(\lambda) = (1/g)(dg/d\lambda) \quad (10.13)$$

Now if  $g(\lambda)$  is chosen such that

$$t(\lambda) = t = \sum \text{Tr } K(s,s) \quad (10.14)$$

then the modified termination condition will be

$$-\frac{dq}{d\lambda} = \oint \text{Tr} [D(s,s;\lambda) - K(s,s)q(\lambda)] \quad (10.15)$$

If power series expansions of  $D(u,s;\lambda)$  and  $q(\lambda)$  are made, the modified recursive characterizer and termination equations become

$$D_n(u,s) = K(u,s)q_n + \oint K(u,v)D_{n-1}(v,s) \quad (10.16)$$

$$-n q_n = \oint \text{Tr} [D_{n-1}(s,s) - K(s,s)q_{n-1}] \quad (10.17)$$

$$q_0 = 1 \quad D_0(u,s) = K(u,s) \quad (10.18)$$

The result is that  $q_1 = 0$  and manipulation with the possibly non-existent  $K(s,s)$  has been avoided. From Eqs. (10.13) and (10.14) there follows that

$$q(\lambda) = e^{t\lambda} p(\lambda) \quad (10.19)$$

so that no new zeros have been introduced into the determinant, both  $q(\lambda)$  and  $D(u,s;\lambda)$  are entire functions, and the modified secular equation

$$q(\lambda) = 0 \quad (10.20)$$

yields the same results as does Eq. (10.4).

A more serious crisis can arise in the previously described method of extraction of normal modes, Eq. (10.5). The trivial zero vector may be obtained if  $c_0$  and  $s_0$  are injudiciously chosen, but, normally such values of  $s_0$  are the exceptional, easily avoided ones. In some cases, however, the process may fail entirely for all values of  $s_0$  and  $c_0$  because  $C(u,s;\lambda_0)$  may vanish identically. This situation may arise if  $\lambda_0$  is a multiple root of the secular equation. The procedure for extracting the normal modes then becomes considerably more complicated.

Since no information was lost in translating the problem from Maxwell's equations to the present formulation, it may yet be expected that some further set of operations may be applied to the characterizer to extricate the complete set of normal modes from the equations even in case  $\lambda_0$  is a repeated root of  $p(\lambda)$ . The appropriate operations for this purpose will now be expounded.

## 11. NORMAL MODES

The present reformulation of Maxwell's equations incorporates all the results obtainable for any linear medium in the corresponding resolvent  $H(u, s; \lambda)$ . The physical properties of the medium are reflected in the analytic properties of this function. In particular, the resonance condition or dispersion relation characterizing the medium is a mathematical statement which locates the poles of the resolvent, while its residues describe the normal modes of the system.

The basis for obtaining the normal modes is the principle that if there be extracted from the resolvent any matrix  $T(u, s; \lambda)$  with the property that

$$T(u, s; \lambda_0) = \lambda_0 \oint K(u, v) T(v, s; \lambda_0) \quad (11.1)$$

where  $\lambda_0$  is a root of the secular equation, then this matrix will generate solutions to the homogeneous equation, Eq. (7.8), as follows.

$$V(u) = \oint T(u, s; \lambda_0) c(s) \quad (11.2)$$

$$I(u) = \oint Y(u, s) V(s) \quad (11.3)$$

Here,  $c(s)$  is an arbitrary three-vector function of one composite index. For this sourceless solution to be nontrivial, it is required merely that the vector  $c(s)$  be not one that is annihilated in summing with  $T(u, s; \lambda_0)$ . This requirement is normally so weak that the summation may be avoided entirely by choosing a unit function for the arbitrary vector:  $c(s) = c_0 l(s, s_0)$ , where  $s_0$  is any value of the composite index and  $c_0$  any constant three-vector, each selected so that the resulting  $V(u)$  does not vanish. This procedure offers no difficulty unless  $T(u, s; \lambda_0)$  is

identically zero.

As indicated previously, the characterizer  $C(u, s; \lambda)$  is such a matrix and it does yield a normal mode for each root of the determinant in this manner. The process fails, however, if  $C(u, s; \lambda)$  vanishes identically at  $\lambda = \lambda_0$ . This can occur if  $\lambda_0$  is a multiple root of the secular equation. Another matrix  $T(u, s; \lambda)$  must then be found which satisfies Eq. (11.1) for this  $\lambda = \lambda_0$  and which does not vanish. Clearly, this corresponds to finding the residue of the resolvent at a pole of higher order. The procedure for finding the normal modes for multiple eigenvalues is, in fact, analogous to that of extracting such a residue and is justified by the following Successor Theorem.

Let a differential operator  $D_n(\lambda)$  be first defined by

$$D_n(\lambda)f(\lambda) = \frac{\lambda^n}{n!} \frac{\partial^n f}{\partial \lambda^n} \quad (11.4)$$

It is readily verified that this operator's effect on a product is expressed by

$$D_n(\lambda) [f(\lambda)g(\lambda)] = \sum_{m=0}^n D_{n-m}(\lambda)f(\lambda) D_m(\lambda)g(\lambda) \quad (11.5)$$

Furthermore, by virtue of the property of the resolvent given in Eq. (7.6), and defining the "powers" of the resolvent just as those of the kernel were defined in Eq. (8.11), it is found that

$$D_n(\lambda)H(u, s; \lambda) = \lambda^n H^{n+1}(u, s; \lambda) \quad (11.6)$$

Let now the successors  $S_n(u, s; \lambda)$  and the terminators  $g_n(\lambda)$  be defined by

$$S_n(u, s; \lambda) = D_n(\lambda)C(u, s; \lambda) \quad g_n(\lambda) = D_n(\lambda)p(\lambda) \quad (11.7)$$

The relation between these quantities is revealed by applying the  $D_n(\lambda)$  operator to the defining equation

$$C(u, s; \lambda) = H(u, s; \lambda) p(\lambda) \quad (11.8)$$

to obtain, by use of Eqs. (11.5) and (11.6),

$$S_n(u, s; \lambda) = \sum_{m=0}^n \lambda^{n-m} H^{n+1-m}(u, s; \lambda) g_m(\lambda) \quad (11.9)$$

Comparing this expression with the same one written for  $S_{n-1}(u, s; \lambda)$ , there is immediately obtained the recursion relation

$$S_n(u, s; \lambda) = \lambda \oint H(u, v; \lambda) S_{n-1}(v, s; \lambda) + H(u, s; \lambda) g_n(\lambda) \quad (11.10)$$

The effect of the kernel  $K(u, s)$  operating on a successor is obtainable from this recursion relation by applying the resolvent equation, Eq. (7.3), directly. The result is

$$S_n(u, s; \lambda) = \lambda \oint K(u, v) S_n(v, s; \lambda) + R_n(u, s; \lambda) \quad (11.11)$$

$$R_n(u, s; \lambda) = \lambda \oint K(u, v) S_{n-1}(v, s; \lambda) + K(u, s) g_n(\lambda) \quad (11.12)$$

Thus, the successor  $S_n(u, s; \lambda)$  will satisfy Eq. (11.1), provided that  $R_n(u, s; \lambda_0)$  vanishes.

To apply this Successor Theorem to the problem of calculating the normal modes of a system, consider a case of a determinant  $p(\lambda)$  which has a zero of multiplicity  $z$  at  $\lambda = \lambda_0$ ; that is,  $p(\lambda)$  has  $(1 - \lambda/\lambda_0)^z$  as a factor. It may then be that  $C(u, s; \lambda)$  contains  $(1 - \lambda/\lambda_0)^e$  as a factor, with  $0 \leq e < z$ . If  $e \neq 0$ ,  $C(u, s; \lambda_0)$  vanishes and no normal modes are obtainable from the characterizer. But the appropriate successor may then be used in lieu of the characterizer to generate a sourceless solution. For, in this case,  $g_n(\lambda_0) = 0$  for  $0 \leq n < z$  while  $g_z(\lambda_0) \neq 0$  and  $S_n(u, s; \lambda_0) = 0$  for  $0 \leq n < e$  while  $S_e(u, s; \lambda_0) \neq 0$ . This implies that  $R_e(u, s; \lambda_0) = 0$ , so that  $S_e(u, s; \lambda_0)$  satisfies Eq. (11.1) and yields a solution to the homogeneous equation.

While this disposes of the problem of obtaining a normal mode at any eigenfrequency, it leaves the question of whether any other normal modes, independent of the one obtained as above, may exist for the same frequency. The rather complex answer to this question may best be derived from a detailed study of the structure of the resolvent for a degenerate kernel, which will now be outlined.

In the degenerate case, all sourceless solutions are of the form

$$V(s) = A(s) X \quad (1 - \lambda_0 R) X = 0 \quad (11.13)$$

Hence, the question reduces to that of determining the number of independent eigenvectors  $X$  of the obverse matrix  $R$  for  $\lambda = \lambda_0$ . This question, together with the main problem of how to extract these normal modes from the kernel, may be resolved by examining the Jordan canonical form of the obverse.<sup>21,22</sup> If  $S$  is the similarity transformation matrix which reduces the obverse to its canonical form, it becomes clear from an examination of the detailed structure of the matrices involved that the normal modes sought are linear combinations of certain specific columns of the matrix  $A(s)S$ , the eigencolumns. While this matrix is beyond calculation, the internal structure of the successors is such that they too are formed from linear combinations of certain columns of this matrix. The successors for a degenerate kernel are, from Eq. (9.1),  $S_n(u, s; \lambda) = A(u) D_n(\lambda) Q(\lambda) B(s)$  and upon examining the canonical form of  $D_n(\lambda) Q(\lambda)$ , it is found in general that, first, for  $e \leq n < z$ , each  $S_n(u, s; \lambda_0)$  is formed from linear combinations of columns of  $A(u)S$  which include some eigencolumns; second,  $S_e(u, s; \lambda_0)$  is composed of only eigencolumns but the other successors include extraneous columns; third, eigencolumns not

included among the first few nonvanishing successors will be found among the remaining successors; finally, all the eigencolumns are represented among the complete set of successors  $S_n(u, s; \lambda_0)$  for  $e \leq n < z$ .

Being thus assured that all the independent normal modes are obtainable from linear combinations of the several successors, there remains only to select the appropriate linear combinations to extract only the eigencolumns, filtering out the extraneous columns, and yet obtaining all the independent modes. The Successor Theorem shows how to accomplish this.

First, as already demonstrated, at least one normal mode may be obtained from  $S_e(u, s; \lambda_0)$  as  $V_1(u) = \oint S_e(u, s; \lambda_0) c_1(s)$ . If this successor incorporates more than one eigencolumn, then these other modes will be obtainable as well by simply choosing other arbitrary vectors  $c_m(s)$  to form new, independent linear combinations of the eigencolumns in  $S_e(u, s; \lambda_0)$  in the same way.

Next, once all the independent modes within  $S_e(u, s; \lambda_0)$  have been exhausted, the eigencolumns in  $S_{e+1}(u, s; \lambda_0)$  may be weeded out in a similar manner, except that the selection of arbitrary vectors  $c_m(s)$  must be severely restricted to avoid linear combinations in which the extraneous columns appear. From Eqs. (11.11) and (11.12), it is clearly sufficient that these  $c_m(s)$  come from just that set of vectors which was excluded in the previous step; that is, these new vectors must be selected from among all those which are annihilated by  $S_e(u, s; \lambda_0)$ , for then  $\oint R_{e+1}(u, s; \lambda_0) c_m(s)$  vanishes and a new set of normal modes arises from  $\oint S_{e+1}(u, s; \lambda_0) c_m(s)$ . More modes, in fact all of them, will be similarly obtainable from the other successors  $S_n(u, s; \lambda_0)$  by summing with vectors



$c_m(s)$  which are annihilated by  $S_{n-1}(u, s; \lambda_0)$  and not by  $S_n(u, s; \lambda_0)$  but which are otherwise arbitrary. The process must terminate no later than when  $S_{z-1}(u, s; \lambda_0)$  has been forced to yield its eigencolumns.

Much of the vagueness associated with this process may be eliminated through a system of tabulation which traces the progress towards the extraction of all the normal modes from the successors, provides upper bounds on the number of modes which each successor may be expected to yield, and signals the end of the process as soon as all the modes have been obtained, thereby averting a fruitless search for non-existent modes. The tabulation stems from a classification of all possible structures which any matrix may have in its Jordan canonical form. With the eigenvalue  $\lambda_0$ , of multiplicity  $z$ , of the obverse matrix  $R$  there is associated a classifier,  $h$ , consisting of  $z$  positive integers,  $h_m$ , with the properties

$$\sum_{m=1}^z h_m = z \quad (11.14)$$

$$h_m \geq h_{m+1} \quad (11.15)$$

There are as many independent eigenvectors of  $R$ , or eigencolumns of  $A(u)S$ , or normal modes, as there are nonzero elements of  $h$ . To a mode obtained from  $S_n(u, s; \lambda_0)$  there corresponds a nonzero element of  $h$  of value  $z-n$ . These conditions jointly delimit the number of steps required in the process of extraction of the solutions to the homogeneous equation from the successors.

With the observation that the terminators and successors can be calculated directly from the determinant and characterizer coefficients, found from Eqs. (10.6)-(10.8), as

$$g_n(\lambda) = \sum_{m=n}^{\infty} \binom{m}{n} p_m \lambda^m \quad S_n(u, s; \lambda) = \sum_{m=n}^{\infty} \binom{m}{n} c_m(u, s) \lambda^m \quad (11.16)$$

the foregoing is seen to constitute a complete program for obtaining all the normal modes of a given linear medium.

## 12. GENERAL SOLUTIONS

The theory presented herein not only reformulates the electromagnetics problem of a linear medium in terms of a new equation to replace those of Maxwell but prescribes a step-by-step procedure for solving this resolvent equation and for extracting all the pertinent information about the system as well. The information obtainable from this formulation excludes none that could be obtained from Maxwell's equations, but it emphasizes the dispersion, resonance, or other existence condition characteristic of the medium and the complete set of normal modes associated with it, rather than the actual fields produced by any distribution of sources and the power and energy carried by the fields. The former are actually the relations of greatest interest and significance in most situations.

The recursive solution for these quantities embodied in Eqs. (10.6)-(10.8) demonstrates the possibility of obtaining a complete, exact solution to the resolvent equation. This particular method of solution may, however, be among the least efficient of the many which are available. The resolvent equation

$$H(u,s;\lambda) = K(u,s) + \lambda \int K(u,v)H(v,s;\lambda) \quad (12.1)$$

is the result of central importance in the reformulation and should be considered on its own merits, without any particular solution procedure appended to it. A host of analytical tools may be applied for its solution, for which purpose a compilation of the equation's main properties and a few approaches to its analysis should be useful.

The Neumann series solution to Eq. (12.1), obtained directly by iteration, has been given in Eq. (8.10). As already noted, this expansion is convergent only for small  $\lambda$  and cannot be used to obtain the normal modes. It should be useful for high-frequency approximations. Furthermore, various analytic continuation techniques<sup>12</sup> may be applied to this series to extend its validity to the entire frequency spectrum. The Fredholm solution, Eqs. (10.1)-(10.3), has been designed at the outset for validity at all frequencies, both the characterizer and determinant being entire functions. In this formulation, the existence conditions are expressed by the vanishing of the determinant.

For special forms of the kernel, a complete, exact, closed solution is immediately obtainable. One such type of kernel is the degenerate one, which was treated at length in the foregoing. Its resolvent is given exactly by Eq. (7.13) and the existence conditions are just those which an eigenvalue of the obverse matrix must satisfy. Another type of solvable kernel, which may be termed "ideal," is one of the form

$$K(u,s) = N(s) l(u,s) \quad (12.2)$$

The unit function permits a direct solution to the resolvent equation. The resolvent of the ideal kernel is found to be

$$H(u,s;\lambda) = [1 - \lambda N(s)]^{-1} N(s) l(u,s) \quad (12.3)$$

and the existence condition is

$$\det [1 - \lambda N(s)] = 0 \quad (12.4)$$

It should be noted that the composite index appears here explicitly in the secular equation. Linear combinations of degenerate and ideal kernels also lead to closed forms of resolvents, with the aid of some other

properties of resolvents listed below.

The differential equation satisfied by the resolvent is

$$\frac{\partial H(u, s; \lambda)}{\partial \lambda} = \int H(u, v; \lambda) H(v, s; \lambda) \quad (12.5)$$

With the initial condition

$$H(u, s; 0) = K(u, s) \quad (12.6)$$

this may be considered an alternate definition of the resolvent. More basic still is the functional equation for the resolvent,

$$H(u, s; \lambda) - H(u, s; \xi) = (\lambda - \xi) \int H(u, v; \lambda) H(v, s; \xi) \quad (12.7)$$

which identifies the class of functions to which all resolvents belong.

General properties of resolvents include that of commutation:

$$\int H(u, v; \lambda) H(v, s; \xi) = \int H(u, v; \xi) H(v, s; \lambda) \quad (12.8)$$

In particular, the resolvent commutes with the kernel. The resolvent of a kernel which is proportional to one whose resolvent is known is found as

$$K(u, s) = a K_0(u, s) \quad H(u, s; \lambda) = a H_0(u, s; a\lambda) \quad (12.9)$$

If two kernels are orthogonal; that is, if

$$\int K_1(u, v) K_2(v, s) = \int K_2(u, v) K_1(v, s) = 0 \quad (12.10)$$

then the resolvent of the sum is the sum of the resolvents:

$$K(u, s) = K_1(u, s) + K_2(u, s) \quad H(u, s; \lambda) = H_1(u, s; \lambda) + H_2(u, s; \lambda) \quad (12.11)$$

The resolvent of a kernel which is the sum of a degenerate one and one whose resolvent is known is expressible in closed form, as follows. If

$$K(u, s) = A(u)B(s) + K_0(u, s) \quad (12.12)$$

then let

$$A_0(u; \lambda) = A(u) + \lambda \int H_0(u, s; \lambda) A(s) \quad (12.13)$$

$$B_0(s;\lambda) = B(s) + \lambda \oint B(u)H_0(u,s;\lambda) \quad (12.14)$$

$$R_0(\lambda) = \oint B(s)A_0(s;\lambda) = \oint B_0(s;\lambda)A(s) \quad (12.15)$$

and the resolvent for this partially degenerate kernel is

$$H(u,s;\lambda) = H_0(u,s;\lambda) + A_0(u;\lambda)[1-\lambda R_0(\lambda)]^{-1}B_0(s;\lambda) \quad (12.16)$$

Finally, it is noteworthy that the recursion relations of Eqs. (10.6)-(10.8), which produce the power series expansions for the characterizer and determinant of the Fredholm solution, are explicitly solvable, yielding expressions for each of the determinant and characterizer coefficients which may be evaluated directly from the kernel. The kernel must first be processed to yield the terminants,

$$t_n = (-1/n) \oint \text{Tr } K^n(s,s) \quad (12.17)$$

In terms of these, the explicit expressions for the determinant and characterizer coefficients are

$$p_n = \sum \prod_{m=1}^n \frac{t_m^{e_m}}{e_m!} \quad (12.18)$$

$$C_n(u,s) = \sum_{m=0}^n K^{m+1}(u,s)p_{n-m} \quad (12.19)$$

In Eq. (12.18), the summation is over all terms of the indicated type which may be formed from  $n$  positive integers  $e_m$  such that  $0 \leq e_m \leq n$  and

$$\sum_{m=1}^n m e_m = n \quad (12.20)$$

Eqs. (12.17) and (12.18) epitomize the entire processing required to extract the existence condition from the kernel. The recursion relations may, however, be more convenient than these cumbersome explicit expressions.

The foregoing brief summary of the analytic properties of resolvents is intended as a guide to the most efficient way of attacking the resolvent equation for any particular problem. It may be seen that the ease with which the equation may be manipulated and solved depends mainly on the form of the kernel. A crucial point to be noted is that the form of the kernel is in great measure dictated by the original choice of the basis of representation for the solution. Although the choice of the complete sets is in principle arbitrary, a judicious initial choice will clearly simplify all further calculations.

### 13. GENERAL TECHNIQUES

The theory developed herein may be extended through general and special techniques applicable in various situations. Specialization may be desirable in certain cases with simplifying features for which the use of the formidable machinery set up here would be an extravagance. Generalizations, on the other hand, can lead to the relaxation of some of the restrictions under which the theory is valid.

One restriction which may easily be dropped is that which requires the constitutive tensors to be independent of frequency. This condition was imposed for convenience of interpretation of the secular equation as a dispersion relation or resonance condition. Many of the equations obtained, particularly the termination condition, are not valid if the kernel is a function of the parameter  $\lambda$ , which is the case when the constitutive tensors depend on the excitation frequency.

To adapt the theory for the case of frequency-dependent constitutive parameters requires only the redefinition of the parameter  $\lambda$  and of the kernel  $K(u,s)$ . Abandoning the original relation between  $\lambda$  and the frequency, expressed in Eq. (6.5), the parameter  $\lambda$  is now to be considered an arbitrary auxiliary variable unrelated to any physical quantity. The kernel  $K(u,s)$  of Eq. (6.6) is now more conveniently redefined as simply

$$K(u,s) = \int Z(u,v)Y(v,s) \quad (13.1)$$

The entire subsequent theory remains valid, with only the following slight modification. With this new definition of the kernel, Eq. (6.8)



represents a problem somewhat more general than required, but one which reduces to that of Maxwell's equations for  $\lambda = 1$ . This last specification is therefore to be introduced into all the equations so as to make them valid for the electromagnetics problem in question. In particular, the new secular equation is

$$p(l) = 0 \quad (13.2)$$

which, despite the appearance of overdetermination, simply fixes the wave number  $k$  in relation to the dimensions and other parameters of the system. Eq. (13.2) can hence be interpreted in the same way as was Eq. (10.4). Complications may arise, however, since the expansions of the determinant and characterizer can no longer be claimed to converge over the entire frequency spectrum, for the frequency dependence of the constitutive parameters may introduce singularities into these functions. Analytic continuation techniques may then be required to suit the particular frequency dependence of the medium.

A more sweeping alteration of the theory is called for in case the kernel is singular. A weak type of singularity has already been disposed of through a modification of the termination condition as in Eq. (10.15). For stronger singularities, however, in which higher powers of the kernel are also singular, this modification is ineffectual. This is the case, for example, with the ideal kernel of Eq. (12.2). What is required in such cases is a redefinition of the determinant, as follows.

$$H(u, s; \lambda) = C(u, s; \lambda) / p(s; \lambda) \quad (13.3)$$

The ideal determinant,  $p(s; \lambda)$ , is now a function of the composite index  $s$  as well as of the parameter  $\lambda$ . The termination condition must correspondingly be modified to

$$-\frac{\partial p(s;\lambda)}{\partial \lambda} = \sum_u \text{Tr } C(u,s;\lambda) \quad (13.4)$$

where, as indicated, the generalized summation is to be performed only over the composite index  $u$ . A similar modification in summation is to be introduced into the appropriate equations of the rest of the theory, which thereupon remains valid even in such singular cases.

The new secular equation for such cases is

$$p(s;\lambda) = 0 \quad (13.5)$$

Its interpretation is no more difficult than that of the previous form. Here, the existence condition relates the wave number  $k$  implicit in  $\lambda$  to not only the dimensions of the system but to the transform index  $s$  as well. This index will then readily be assigned a physical significance, such as that of a propagation vector in a dispersion relation.

Specialization of the theory is called for to take advantage of such simplifying features of a problem as its symmetry properties. For example, a typical propagation problem may involve axial symmetry, which invites the introduction of a propagation factor of the form  $e^{-jbz}$  in cylindrical coordinates. This simplifies the subsequent calculations since this will be the dominant, if not the only, axial variation of the fields. The resulting simplification can be introduced at the outset by modifying the curl operator of Eq. (3.2) to the extent of replacing  $\partial/\partial z$  by  $\partial/\partial z - jb$  and then eliding the propagation factor. Similar modifications may be made in case of circular or other symmetry.

More generally, the presence of symmetry in a problem can lead, with the appropriate choice of the basis of representation, to a reduction in the dimensionality of the resolvent equation. If the composite index

can be partitioned as  $s = (s_1, s_2)$ , so that the kernel is ideal with respect to one part of the index, then the multiple summations can be reduced in number. Thus, if  $K(u, s) = K(s_1; u_2, s_2)l(u_1, s_1)$ , then  $H(u, s; \lambda) = H(s_1; u_2, s_2; \lambda)l(u_1, s_1)$ , with the dimensionally reduced resolvent equation

$$H(s_1; u_2, s_2; \lambda) = K(s_1; u_2, s_2) + \lambda \oint_{v_2} K(s_1; u_2, v_2)H(s_1; v_2, s_2; \lambda) \quad (13.6)$$

in which the status of  $s_1$  is that of a parameter, rather than of a dummy index.

Another feature of some media which allows the calculations to be reduced is the presence of conducting regions or boundaries. These can be treated formally by introducing infinite values of the conductivity into the permittivity dyadic for those values of the position vector which correspond to the perfectly conducting regions of the space. In the case of a closed system bounded by conducting walls, the result is that the integrations involved in the transformations become limited to the interior region of the system.

If approximate solutions suffice for the application at hand, many simplifying techniques are available. A general kernel could be approximated by one which is degenerate, whereupon an exact, closed-form solution for the resolvent for the approximate kernel is obtainable. Another situation in which the full machinery of the theory is needlessly cumbersome if approximate solutions are adequate is one which represents a slight perturbation of a medium for which the kernel is degenerate. The kernel will then be expressible as in Eq. (12.12), in which  $K_0(u, s)$  is, in some appropriate sense, small. The Neumann series, Eq. (8.10), is then a rapidly convergent expression for the resolvent  $H_0(u, s; \lambda)$  for

the perturbing kernel  $K_0(u,s)$ . The solution given in Eq. (12.16) can then be used, with  $H_0(u,s;\lambda)$  known at least approximately. Still another type of ready-made approximation is available in case the kernel is sharply peaked near  $u = s$ , though perhaps not actually of the ideal type. The approximate resolvent is then easily seen to be

$$H(u,s;\lambda) = [1 - \lambda \int_v K(u,v)]^{-1} K(u,s) \quad (13.7)$$

This last approach can be the basis for a method of "moments" for approximating the resolvent, in which the resolvent is expanded in a power series about the value of the composite index at which the kernel is peaked and a set of simultaneous matrix equations is solved for the resolvent.

An extensive generalization of the theory may be undertaken with the goal of lifting the restriction to time-invariant media. While slow time variations of the constitutive parameters present no difficulties in yielding to analysis as modulated waves, rates of variation comparable to those of the fields would require the abandonment of the original steady state, harmonic analysis. The theory could be reformulated in terms of a  $\max_t$  operator which incorporates the partial derivatives with respect to time. The only remaining restriction upon the applicability of the theory would then be simply to media which are linear.

#### 14. ILLUSTRATION

As an illustration of the applicability and feasibility of the method of analysis presented herein, there will now be obtained there-with the solution to a particular class of problems in electromagnetic theory. Although this example is computationally quite trivial and is certainly not intractable to standard methods of analysis, it does aid in concretizing the rather abstract theory and possesses sufficient generality to be of interest. A specific example of this class of problems will also be examined in detail in order to display the nature of the calculations involved in the matrix formulation which underlies the theory.

The system to be analyzed is that of an infinite, homogeneous, linear medium. Such a medium is characterized by permittivity and permeability tensors which are independent of position, though otherwise arbitrary:

$$\epsilon(r) = \epsilon \quad \mu(r) = \mu \quad (14.1)$$

The complete solution, as embodied in the resolvent for this system, will be obtained and, in particular, the dispersion relation for the medium. The theory may be applied as follows.

As there is no characteristic physical dimension of the system and since, moreover, the constitutive tensors may be functions of frequency, the kernel as defined in Eq. (13.1) will be used and  $\lambda$  will be set equal to unity in the final result. For lack of any reason to complicate the calculations, the transformer  $c(r,s)$  which forms the basis

of representation will be assumed to be a scalar. The immittance kernels are then given by

$$Z(u,s) = \oint d(u,r) (k_e)^{-1} \text{curl } c(r,s) \quad (14.2)$$

$$Y(u,s) = \oint d(u,r) (k_\mu)^{-1} \text{curl } c(r,s) \quad (14.3)$$

as is clear from an inspection of Eqs. (6.2), (4.14), (3.7), (3.4), and (3.5). As a result of the constancy of the constitutive tensors and the scalar character of  $d(u,r)$ , these may be rewritten as

$$Z(u,s) = (k_e)^{-1} \oint d(u,r) \text{curl } c(r,s) \quad (14.4)$$

$$Y(u,s) = (k_\mu)^{-1} \oint d(u,r) \text{curl } c(r,s) \quad (14.5)$$

In view of the orthonormality relation of Eq. (4.7), it is clearly advantageous to select  $c(r,s)$  so that the curl operation leaves it intact. If rectangular coordinates are selected, this is easily accomplished, for then the operator of Eq. (3.1) will clearly leave an exponential function unchanged in form. Hence

$$c(r,s) = e^{-jr \cdot s} \quad (14.6)$$

is selected, the composite index  $s$  being considered a vector. This amounts to nothing more than choosing a Fourier transform to effect the solution. The result of operating on  $c(r,s)$  with the curl operator is expressed by

$$\text{curl } c(r,s) = c(r,s)(-j\beta) \quad (14.7)$$

as follows from Eqs. (3.1) and (5.7). Hence,

$$\oint d(u,r) \text{curl } c(r,s) = l(u,s)(-j\beta) \quad (14.8)$$

and the immittance kernels are

$$Z(u,s) = (-j/k) \epsilon^{-1} \oint l(u,s) \quad (14.9)$$

$$Y(u,s) = (-j/k) \mu^{-1} \oint l(u,s) \quad (14.10)$$

The kernel is therefore simply

$$K(u, s) = (-1/k^2) \epsilon^{-1} \beta_\mu^{-1} \beta l(u, s) \quad (14.11)$$

This kernel is of the ideal type, as in Eq. (12.2), with  $N(s) = (-1/k^2) \epsilon^{-1} \beta_\mu^{-1} \beta$ . The resolvent for such kernels is given in Eq. (12.3). In this case, letting

$$D(s) = \epsilon^{-1} \beta_\mu^{-1} \beta \quad (14.12)$$

the resolvent is

$$H(u, s; \lambda) = [1 + (\lambda/k^2) D(s)]^{-1} (-1/k^2) D(s) l(u, s)$$

or, since  $\lambda = 1$ ,

$$H(u, s; 1) = - [k^2 + D(s)]^{-1} D(s) l(u, s) \quad (14.13)$$

and the problem is solved.

The dispersion relation for this medium is given by Eq. (12.4), which may be written in this case

$$\det [k^2 + D(s)] = 0 \quad (14.14)$$

Alternatively, the ideal determinant  $p(s; \lambda)$  may be obtained through the recursion relations as an expansion in powers of  $\lambda$ , which is unity.

What is obtained then is the same as the expansion of Eq. (14.14) by use of Eqs. (9.6) and (9.4). The result is

$$k^6 + \text{Tr}_1 D(s) k^4 + \text{Tr}_2 D(s) k^2 + \text{Tr}_3 D(s) = 0$$

or, since  $\text{Tr}_3 D(s) = \det D(s) = 0$  as a result of the fact that  $\det \beta = 0$ ,

$$k^4 + \text{Tr} D(s) k^2 + \text{Tr}_2 D(s) = 0 \quad (14.15)$$

This is the dispersion relation for any infinite homogeneous medium.

From Eq. (14.6),  $s$  is clearly the propagation vector and the transformation is merely an expansion in plane waves. The normal modes are given by the condition that  $V(s)$  be an eigenvector of the matrix  $D(s)$ .

This disposes of all infinite, homogeneous, linear media, be they anisotropic, dispersive, lossy, with or without electric and magnetic current source distributions of any kind.

A specific example of a medium of this type which is of some interest will illustrate the detailed matrix calculations required by this formulation. Consider a medium which is gyrotropic, both in capacity and in permeability. The former type is realizable in a plasma, the latter in a ferrite. If the preferred direction for both is the z-axis, the constitutive tensors take the form

$$\epsilon = \begin{bmatrix} \epsilon_1 & j\epsilon_2 & 0 \\ -j\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \quad (14.16)$$

The explicit dispersion relation for this medium is desired. From Eq. (14.15), this requires the calculation of the traces of  $D(s) = \epsilon^{-1} \beta \mu^{-1} \beta$ . A direct calculation involves considerable algebraic manipulation, much of which is superfluous. By taking advantage of the structure of the constitutive tensors and of the matrix  $\beta$ , the process is reducible to elementary matrix manipulations, with a spectacular saving in labor, as follows.

The structure of the constitutive dyadics invites a partition of the matrices into transverse and longitudinal parts:

$$\epsilon = \begin{bmatrix} Z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} \quad \mu = \begin{bmatrix} Y^{-1} & 0 \\ 0 & y^{-1} \end{bmatrix} \quad (14.17)$$

where  $Z$  and  $Y$  are  $2 \times 2$  matrices. Defining the numerical matrix  $S$  as



$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (14.18)$$

and noting the properties

$$S^2 = -1 \quad S^+ = -S \quad (14.19)$$

where + here denotes the transpose of the matrix, the inner structure of the partitions can be conveniently exhibited:

$$Z^{-1} = \epsilon_1 - j\epsilon_2 S \quad z^{-1} = \epsilon_3 \quad Y^{-1} = \mu_1 - j\mu_2 S \quad y^{-1} = \mu_3 \quad (14.20)$$

It follows from Eq. (14.19) that the inverses partition with the same structure:

$$Z = z_1 + z_2 S \quad Y = y_1 + y_2 S \quad (14.21)$$

from which it is clear that Z and Y commute with each other and with S. This simple observation in itself leads to a considerable saving in labor.

The propagation vector s can also be partitioned into transverse and longitudinal parts so that the structure of the  $\beta$  matrix may also be revealed:

$$s = \begin{bmatrix} b \\ c \end{bmatrix} \quad \beta = \begin{bmatrix} cS & -Sb \\ -b^+S & 0 \end{bmatrix} \quad (14.22)$$

where b is a 2 x 1 column and c is a single element. Hence, D(s) is to be obtained as

$$D(s) = \begin{bmatrix} Z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} cS & -Sb \\ -b^+S & 0 \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} cS & -Sb \\ -b^+S & 0 \end{bmatrix} \quad (14.23)$$

which will have the structure

$$D(s) = \begin{bmatrix} D_0 & d_1 \\ d_2 & d \end{bmatrix} \quad (14.24)$$

The traces of this matrix are expressible in terms of the partitions as

$$\text{Tr } D(s) = d + \text{Tr } D_0 \quad (14.25)$$

$$\text{Tr}_2 D(s) = \det D_0 + d \text{Tr } D_0 - d_2 d_1 \quad (14.26)$$

Even the calculation of  $\det D_0$  in Eq. (14.26) can be avoided by recalling that  $\det D(s)$  is known to be zero:

$$\det D(s) = 0 = d \det D_0 + d_2 S D_0^+ S d_1 \quad (14.27)$$

Hence,  $\text{Tr}_2 D(s)$  can be expressed as

$$\text{Tr}_2 D(s) = (-1/d) d_2 S D_0^+ S d_1 + d \text{Tr } D_0 - d_2 d_1 \quad (14.28)$$

which involves nothing more difficult than matrix multiplication.

Now the partitions of  $D(s)$  are immediately obtainable from Eq. (14.23):

$$D_0 = c^2 ZSYS + yZSbb^+S = -c^2 ZY + ySZbb^+S \quad (14.29)$$

$$d = zb^+SYSb = -zb^+Yb \quad (14.30)$$

$$d_1 = -cZSYSb = cZYb \quad (14.31)$$

$$d_2 = -zcb^+SYS = zcb^+Y \quad (14.32)$$

where the commutation relations noted above and the properties of Eq.

(14.19) have been used to effect the simplifications. The slight labor involved in evaluating Eqs. (14.25) and (14.28) can be reduced still further by noting that  $b^+Sb = 0$  since  $S$  is antisymmetric, that  $\text{Tr } S = 0$  and  $\text{Tr } 1 = 2$ , that the trace is a linear operator, and that matrices may be commuted in taking traces, as in Eq. (9.3). Making use of Eq.

(14.21), there is obtained, with little effort,

$$\text{Tr } D(s) = 2c^2(z_2 y_2 - z_1 y_1) - b^2(zy_1 + yz_1) \quad (14.33)$$

$$\begin{aligned} \text{Tr}_2 D(s) = & c^4(y_1^2 + y_2^2)(z_1^2 + z_2^2) + b^4 zy_1 y_1 + \\ & c^2 b^2 [yy_1(z_1^2 + z_2^2) + zz_1(y_1^2 + y_2^2)] \end{aligned} \quad (14.34)$$

In these expressions,  $b^2 = b^+ b$  is the square of the magnitude of the transverse part of the propagation vector. The traces can easily be rewritten in terms of the elements of the constitutive tensors by noting that  $z_1^2 + z_2^2 = \det Z = 1/\det Z^{-1} = 1/(\epsilon_1^2 - \epsilon_2^2)$ , with a similar expression involving  $Y$ .

Introducing these traces into Eq. (14.15) gives the dispersion relation for the doubly gyrotropic medium. Its physical significance becomes more apparent when it is written in terms of the magnitude,  $\beta$ , of the propagation vector  $s$  and of the angle,  $\theta$ , between the direction of propagation and the preferred direction in space.

$$\beta^2 = b^2 + c^2 \quad c^2 = \beta^2 \cos^2 \theta \quad b^2 = \beta^2 \sin^2 \theta \quad (14.35)$$

In terms of a refractive index  $n = \beta/k$ , the dispersion relation can be written after a little rearrangement as

$$A n^4 - B n^2 + C = 0 \quad (14.36)$$

with the coefficients given by

$$A = (\epsilon_1 \sin^2 \theta + \epsilon_3 \cos^2 \theta) (\mu_1 \sin^2 \theta + \mu_3 \cos^2 \theta) \quad (14.37)$$

$$\begin{aligned} B = & 2\epsilon_3 \mu_3 (\epsilon_1 \mu_1 + \epsilon_2 \mu_2) \cos^2 \theta + \\ & [\mu_1 \mu_3 (\epsilon_1^2 - \epsilon_2^2) + \epsilon_1 \epsilon_3 (\mu_1^2 - \mu_2^2)] \sin^2 \theta \end{aligned} \quad (14.38)$$

$$C = \epsilon_3 \mu_3 (\epsilon_1^2 - \epsilon_2^2) (\mu_1^2 - \mu_2^2) \quad (14.39)$$

Eq. (14.36) is a quadratic in  $n^2$ , which may readily be solved to yield the refractive index in any direction.

The homogeneity of the space makes this illustration a trivial application of the theory, the full power and generality of which is only barely in evidence in this example. Here the integrations were so elementary as to reduce the calculations to little more than simple algebra. In general inhomogeneous media the quadratures may be inordinately numerous and cumbersome but the mechanics of setting them up and combining them to extract the desired results are much the same as in this simple illustration.

## 15. CONCLUSIONS

The formalism developed in this work attempts to provide a new vantage point from which to view and attack a large class of problems in electromagnetic theory. Concentrating on the invariable, characteristic features of general linear media and subordinating their less significant details, it unifies the approach to a wide variety of such media, to which the various standard, classical modes of attack have assigned a variety of epithets -- inhomogeneous, anisotropic, etc. The method of analysis reformulates Maxwell's equations, provides an alternate starting point for the extraction of all significant information from a given system. Recognizing that a full description of the medium must implicitly contain all the relevant information about the system, the formulation seeks a direct route from an initial mathematical description of the electrical constitution of the space to all the information about the consequent phenomena which may be of interest and significance.

The present reformulation combines many disparate analytical techniques of great power -- those of partial differential equations, of matrix algebra and calculus, of abstract linear operators, of generalized transforms, and of integral equations -- which may be brought to bear upon the general problem. Acknowledging the self-defeating features of seeking closed-form solutions for media with any but the simplest structures, the formalism allows considerable freedom of choice of the form in which the results are represented. Care has been taken to preserve intact the informational content of Maxwell's equations in effecting the translation to the new language.

Besides providing a single new equation of radically different form to be analyzed for the solution of a problem, this work presents a host of mathematical properties of this equation which may be used to effect its solution or to derive qualitative features of the results. The method is crowned by the prescription of an explicit sequence of mathematical operations which, in principle, lead ultimately to the complete solution for any given linear medium. This formal solution includes the fundamental existence condition characteristic of the medium, the complete set of normal modes for the system, the entire electromagnetic field pattern in response to any given distribution of electric and magnetic current sources, and the power flow and energy distribution accompanying the fields -- in short, all the information which is latent in the description of the medium and excitations. In effect, there is set up a fictive machine which incorporates the physical mechanism of interaction of electromagnetic fields and sources, has as its input a full pointwise description of the electrical content of the space, and produces as its output all the desired results describing the phenomena associated with the medium. Effectively, the formal solution unifies and systematizes the analysis of a general class of problems in electromagnetic theory and reduces such problems to quadratures.

The objection may be raised, and a formidable one it may well be, that the quadratures involved in the process are of such complexity and multiplicity as to be prohibitive. In the same vein, the criticism may be made that the convergence of the expansions for the determinant and characterizer may be so slow as to require an excessive number of terms in the series, with all the calculations which that entails. These

objections may be palliated by noting that the series expansion method is only one of many modes of attacking the resolvent equation and is generally not the most efficient one. Furthermore, the complexity of the computations depends on the choice of the basis of representation for the results. All possible a priori knowledge of the form of the solution should hence be brought to bear upon the choice of basis functions. If some members of the complete set of expansion functions resemble the functions to be expressed, the number of significant terms required in the expansions will be small, though possibly at the cost of complicating the integrations and summations involved. In practice, some compromise will be arrived at between rapidity of convergence and complexity of calculations. The dominant role which can be played by automatic computers in this connection should be noted.

Elaborations and refinements of the theory may take several directions. The formalism could be expanded to handle rapidly time-varying and nonstationary media in a unified way. In fact, a relativistically invariant formulation could be developed by operating on four-tensors instead of three-vectors. Although the equations governing such cases are of a form quite different from that of the Maxwell equations treated here, the basic ideas used in this work are evidently applicable to any set of linear partial differential equations, whatever their domain of definition. Toward the improvement of the efficiency of the method, studies could be undertaken of means of estimating the remainders of the series appearing in the resolvent when these are truncated, of a decision procedure for optimizing the choice of the basis of representation, and of general methods of improving the convergence of

the series representation of the results.

Valuable applications of the theory may be made in the analysis of various inhomogeneous and anisotropic media such as that of a plasma. Ultimately, this work may lead to successful attacks upon such fundamental problems as the formulation of the conditions for the appearance of certain wave phenomena and the development of synthesis procedures for the implementation of desired dispersion relations.



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<p>operations to be applied to the electrical parameters of the medium to yield the characteristic existence conditions, the set of normal modes, and the electromagnetic fields in response to given sources. The results are applicable to time-invariant, linear media which may be inhomogeneous, anisotropic, nonuniform, dissipative, dispersive, with any source distribution.</p> <p style="text-align: center;">○</p>	
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